The Reduced Membrane Equation

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Geometry, PDE and Mathematical Physics Seminar

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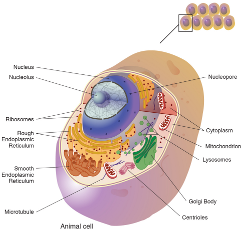
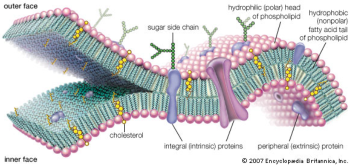
Modeling Biological Membranes





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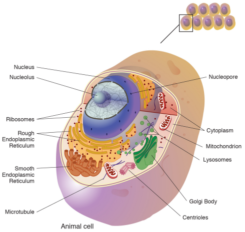

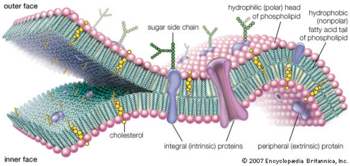

W. Helfrich (1973) suggested to study the critical points of

$$\mathcal{H}[\Sigma] := \int_\Sigma \left(a\,[H + c_o]^2 + bK \right) d\Sigma \,,$$

to model biological membranes.

The Helfrich Energy

Let Σ be a compact (with or without boundary) surface. For an embedding $X : \Sigma \to \mathbb{R}^3$ the Helfrich energy is given by

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- The bending rigidity: $a > 0$.
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Gauss-Bonnet Theorem

The total Gaussian curvature term only affects the boundary.

Euler-Lagrange Equation

The Euler-Lagrange equation associated to $\mathcal{H}$ is

$$\Delta(H + c_o) + 2\,(H + c_o)\,(H\,[H - c_o] - K) = 0\,,$$

a fourth order nonlinear elliptic PDE.

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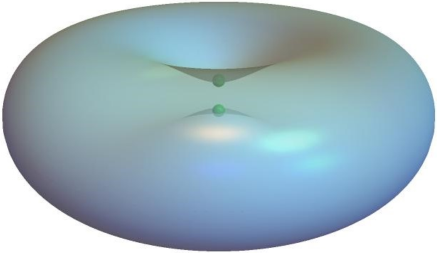
3. Circular Biconcave Discoids with $H^2 - K = c_o^2$. (Far from the axis of rotation.)

Circular Biconcave Discoids



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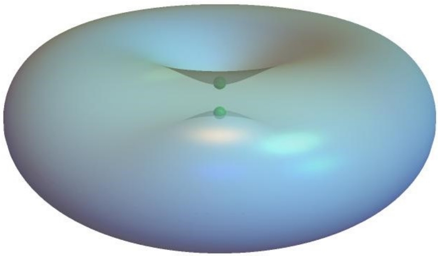

Proposition (López, Palmer & P., Preprint)

Let $\psi \in \mathcal{C}_o^\infty(\Sigma)$ and consider normal variations $\delta X = \psi\nu$, then

$$\delta\mathcal{H}[\Sigma] = 8\pi c_o \psi_{|_{r=0}} .$$

Axially Symmetric Solutions

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Theorem (Palmer & P., 2022)

An axially symmetric disc critical for $\mathcal{H}$ must be:

(I) A planar disc ($H \equiv -c_o = 0$).

(II) A spherical cap ($H \equiv -c_o \neq 0$).

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- The surface must be a topological disc. Annular domains in circular biconcave discoids are critical for $\mathcal{H}$.

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5. Since the surface is regular, our solutions are regular (hence, their derivatives at the cut with the axis of rotation are zero).
6. In conclusion, they are multiples of each other:

$$A(H + c_o) = (H + c_o)z + \nu_3 \,.$$

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1. Taking variations $\delta X = E_3$, we compute the flux formula

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$$r_1 (H + c_o)^2 \partial_n (\star) + r_2 (H + c_o)^2 \partial_n (\star) = 0 \,.$$

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4. Since Σ is a topological disc, we can let $r_2 \to 0$.
5. Hence, $H + c_o \equiv 0$, or $(\star) = A$ holds.

Reduced Membrane Equation

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The reduced membrane equation is the Euler-Lagrange equation for

$$\mathcal{G}[\Sigma] := \int_\Sigma \frac{1}{z^2}\, d\Sigma - 2c_o \int_\Omega \frac{1}{z^2}\, dV = \widetilde{\mathcal{A}}[\Sigma] - 2c_o \int_\Omega |z|\, d\widetilde{V}\,.$$

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Solutions can be viewed as:

- Capillary surfaces with constant gravity in $\mathbb{H}^3$.
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- The right cylinders over elastic curves satisfy the reduced membrane equation.

Modified (Conformal) Gauss Map

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For a real constant c_o we define the map $Y^{c_o} : \Sigma \to \mathbb{S}^4_1 \subset \mathbb{E}^5_1$ by

$$Y^{c_o} := (H + c_o)\,\underline{X} + (\nu, q, q),$$

where $q := X \cdot \nu$ is the support function and

$$\underline{X} := \left(X, \frac{X^2 - 1}{2}, \frac{X^2 + 1}{2} \right).$$

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Theorem (Palmer & P., 2022)

The immersion $X : \Sigma \to \mathbb{R}^3$ is critical for the Helfrich energy $\mathcal{H}$ with respect to compactly supported variations if and only if

$$\Delta Y^{c_o} + \|dY^{c_o}\|^2 Y^{c_o} = 2c_o(0,0,0,1,1)^T.$$

(The map Y^{c_o} fails to be an immersion where $H^2 - K = c_o^2$.)

Special Solutions

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3. Case $\omega := (0, 0, 1, 0, 0)$ is a spacelike vector. Then,

$$H + c_o = -\frac{\nu_3}{z} .$$

(The Reduced Membrane Equation.)

Second Variation Formula

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Theorem (Palmer & P., 2024)

Let $X : \Sigma \to \mathbb{R}^3$ be an immersion critical for the Helfrich energy $\mathcal{H}$ satisfying the reduced membrane equation. Then, for every $f \in \mathcal{C}_o^\infty(\Sigma)$ and normal variations $\delta X = f\nu$,

$$\delta^2\mathcal{H}[\Sigma] = \int_\Sigma f\, F[f]\, d\Sigma + \frac{1}{2}\int_{\partial\Sigma} L[f]\,\partial_n f\, ds\,,$$

where

$$F[f] := \frac{1}{2}\left(P^* + \frac{2}{z^2}\right)\circ P[f]\,.$$

(Here, P is the operator arising as twice the variation of the quantity $H + \nu_3/z$, P^* is its adjoint operator, and L comes from twice the variation of H.)

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- Compute the second variation through the flux formula.

The Operator P as a Jacobi Operator

If $z \neq 0$ everywhere on the surface,

$$\delta^2 \mathcal{H}[\Sigma] = \frac{1}{2} \int_\Sigma P[f] \left(P + \frac{2}{z^2} \right) [f] \, d\Sigma + \oint_{\partial\Sigma} (\partial_n f)^2 \frac{\partial_n z}{z} \, ds \, .$$

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Proposition (Palmer & P., 2024)

Let $X : \Sigma \to \mathbb{R}^3$ be an immersion satisfying the reduced membrane equation. Then, for every $f \in \mathcal{C}_o^\infty(\Sigma)$ and admissible normal variations $\delta X = f\nu$,

$$\delta^2 \mathcal{G}[\Sigma] = -\int_\Sigma \frac{f \, P[f]}{z^2} \, d\Sigma \, .$$

Admissible variations are those that preserve the (hyperbolic) gravitational potential energy.

Symmetry Breaking Bifurcation

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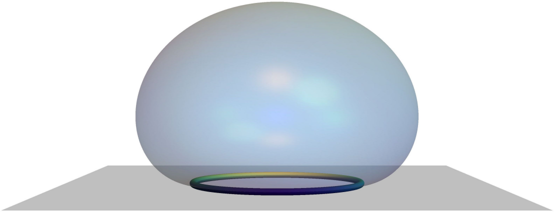

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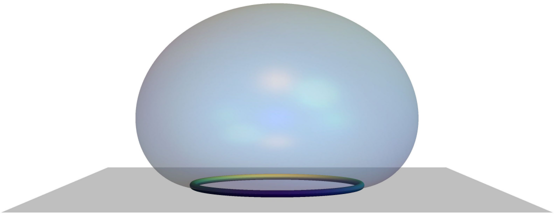

Theorem (Palmer & P., 2024)

Above surface Σ_0 is embedded in a one parameter family of axially symmetric solutions of the reduced membrane equation (parameterized by c_o) which all share the same boundary circle.

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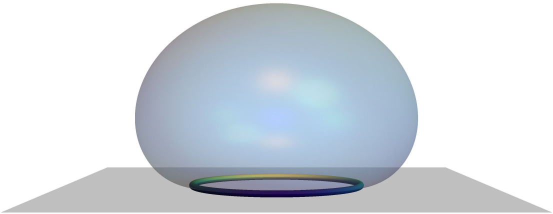
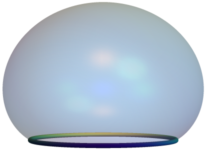


Theorem (Palmer & P., 2024)

Above surface Σ_0 is embedded in a one parameter family of axially symmetric solutions of the reduced membrane equation (parameterized by c_o) which all share the same boundary circle. Precisely, at Σ_0, a non-axially symmetric branch bifurcates.

Axially Symmetric Family



Theorem (Palmer & P., 2024)

Subdomains of Σ_0 are stable and superdomains of Σ_0 are unstable for the functional $\mathcal{G}$.

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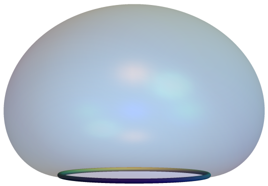

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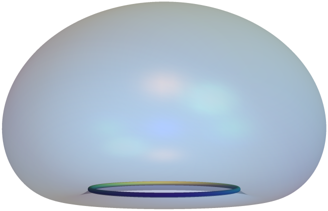

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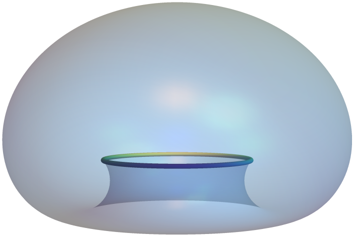

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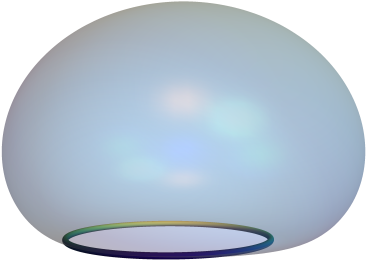
Bifurcating Branch



Conjecture

It is a subcritical pitchfork bifurcation.

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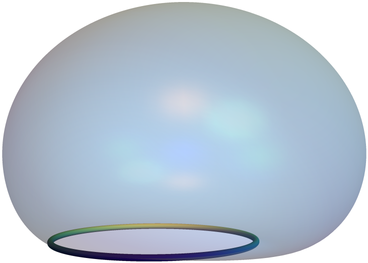

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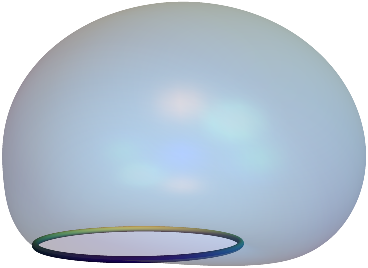

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Applications to Closed Helfrich Surfaces

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Theorem (López, Palmer & P., Preprint)

Let Σ be a closed surface and $X : \Sigma \longrightarrow \mathbb{R}^3$ a $\mathcal{C}^3$ immersion satisfying the reduced membrane equation. Then, $X(\Sigma)$ is a Helfrich surface which intersects the plane $\{z = 0\}$ orthogonally in geodesic circles.

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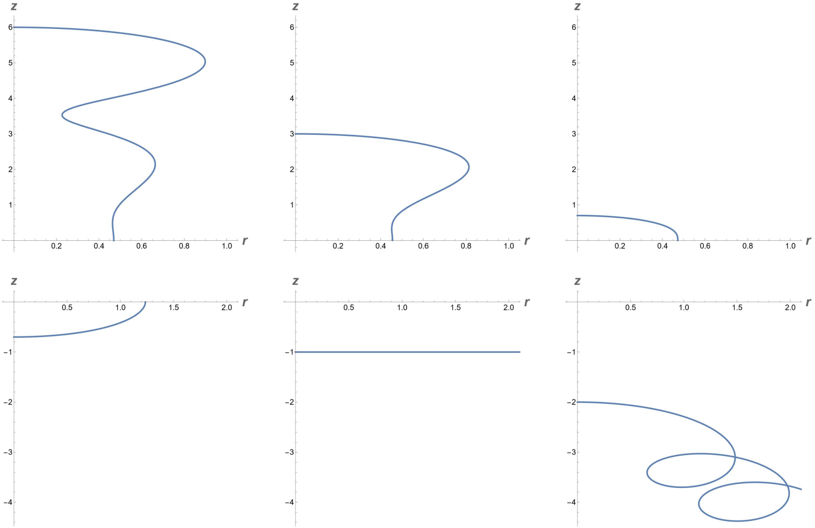
If, in addition, $\partial_n H$ is constant along any connected component of $X(\Sigma) \cap \{z = 0\}$, then the surface is axially symmetric.

Profile Curves



Axially Symmetric Helfrich Topological Spheres

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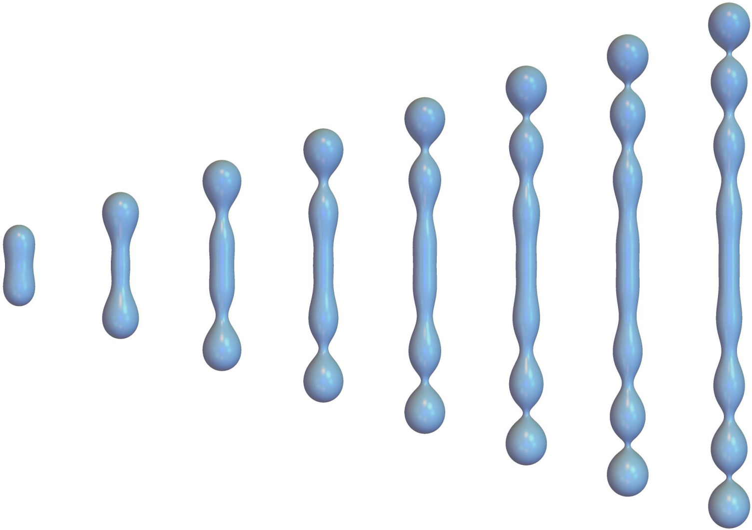
- On the top part (and bottom), ν_3 has at least one change of sign.

First Surfaces in the Family



THE END

- B. Palmer and A. Pámpano, Minimizing Configurations for Elastic Surface Energies with Elastic Boundaries, *J. Nonlinear Sci.* **31-1** (2021), 23.
- B. Palmer and A. Pámpano, The Euler-Helfrich Functional, *Calc. Var. Partial Differ. Equ.* **61** (2022), 79.
- B. Palmer and A. Pámpano, Symmetry Breaking Bifurcation of Membranes with Boundary, *Nonlinear Anal.* **238** (2024), 113393.
- B. Palmer and A. Pámpano, Stability of Membranes, *J. Geom. Anal.*, **34** (2024), 328.
- R. López, B. Palmer and A. Pámpano, Axially Symmetric Helfrich Spheres, *Preprint*, ArXiv: 2501.15668 [math.DG].

Thank You!