The Euler-Helfrich Variational Problem

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Variational Problems for Surfaces

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- W. Helfrich: The Helfrich energy

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Physical Process

Model lipid bilayers formed from a double layer of phospholipids (a hydrophilic head and a hydrophobic tail). These membranes tend to close.

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- The Euler-Helfrich Problem. The boundary components of $\partial\Sigma$ are elastic.

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For an embedding $X : \Sigma \to \mathbb{R}^3$ we consider the total energy

$$E[\Sigma] := \int_\Sigma \left(a\,[H + c_o]^2 + bK \right) d\Sigma + \oint_{\partial\Sigma} \left(\alpha\kappa^2 + \beta \right) ds\,,$$

where $a > 0$, $b \in \mathbb{R}$, $\alpha > 0$ and $\beta \in \mathbb{R}$.

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Rescaling

Let $X : \Sigma \to \mathbb{R}^3$ be critical for E. Then,

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In particular, if $H + c_o \equiv 0$ holds, $\beta > 0$.

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where

$$J := 2\alpha T'' + \left(3\alpha\kappa^2 - \beta\right) T$$

is the Noether current associated to translational invariance of elastic curves.

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Boundary Curves

Let $X : \Sigma \to \mathbb{R}^3$ be an equilibrium with $H + c_o \equiv 0$. Then, each boundary component C is a simple and closed critical curve for

$$F[C] \equiv F_{\mu,\lambda}[C] := \int_C \left([\kappa + \mu]^2 + \lambda \right) ds\,,$$

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where $\mu := \pm b/(2\alpha)$ and $\lambda := \beta/\alpha - \mu^2$.

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- Nitsche's argument involving the Hopf differential.

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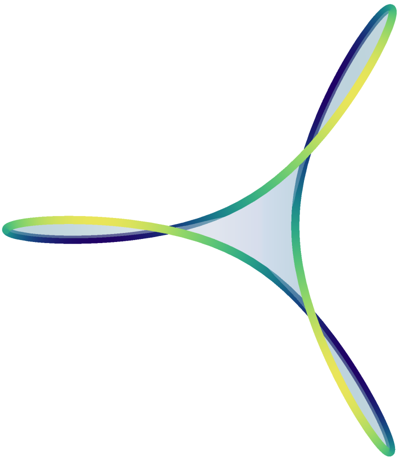

FIGURE: Minimal Surface Spanned by $G(3,1)$.

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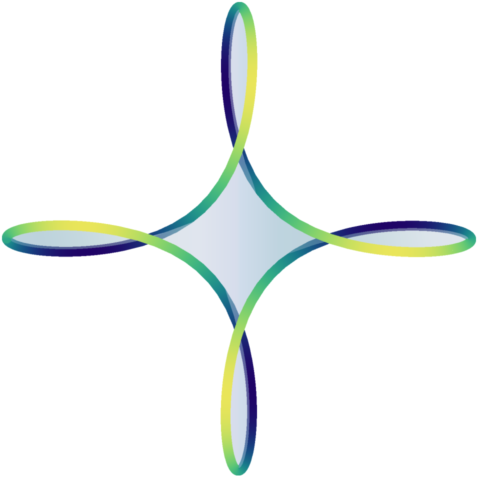

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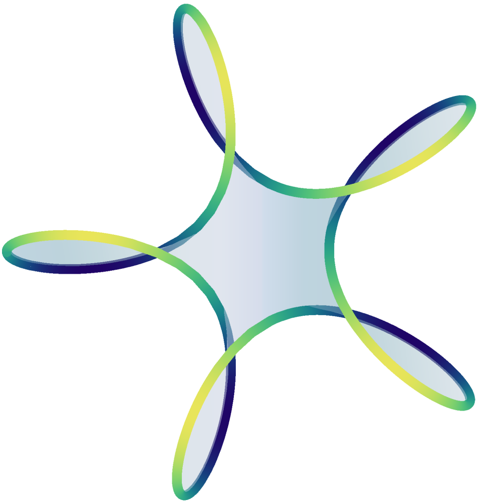

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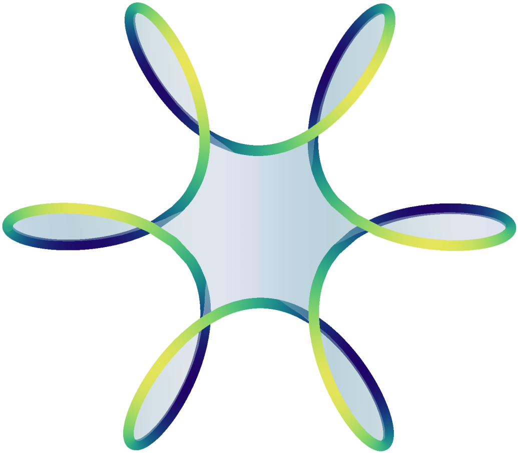

FIGURE: Minimal Surface Spanned by $G(6, 1)$.

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- The boundary knots may not be unknotted, although they are of the same type.
- Algorithm based on the mean curvature flow for fixed boundary.

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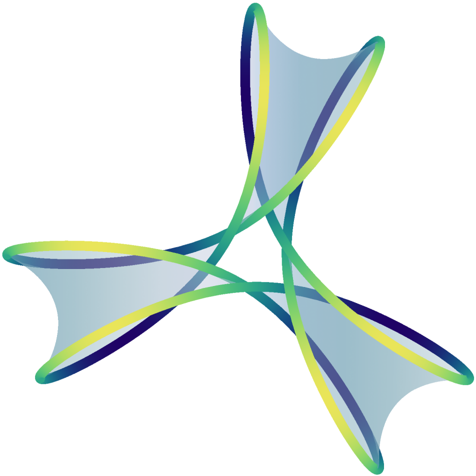

FIGURE: Minimal Surface Spanned by Two $G(3,1)$.

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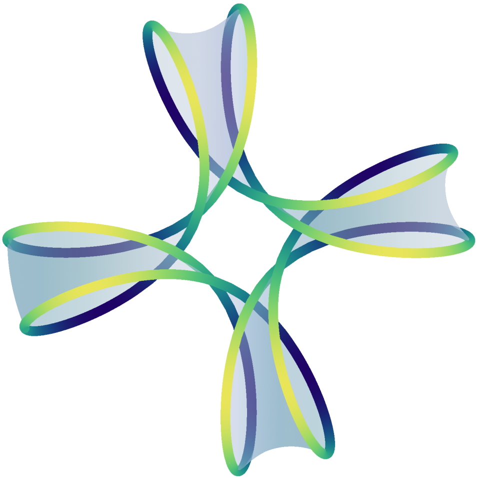

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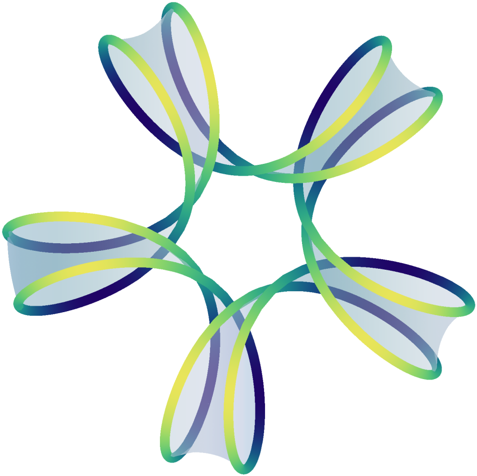

FIGURE: Minimal Surface Spanned by Two $G(5,1)$.

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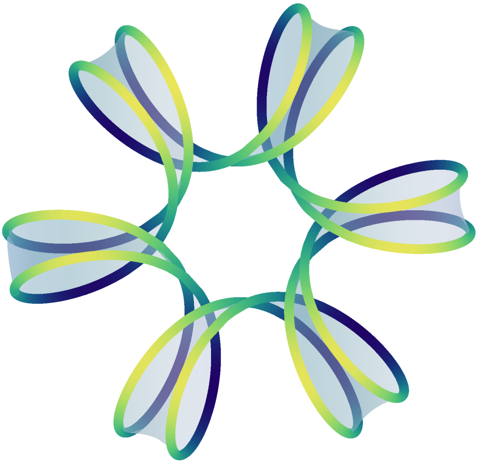

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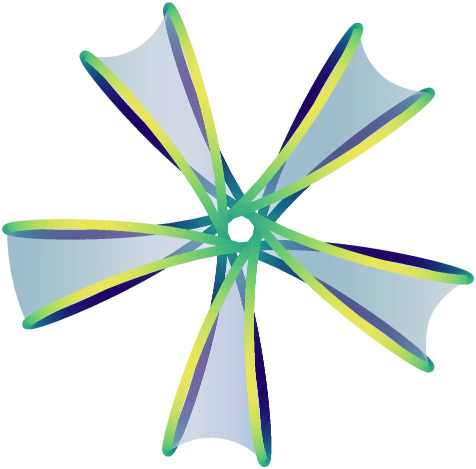

FIGURE: Minimal Surface Spanned by Two $G(5, 2)$.

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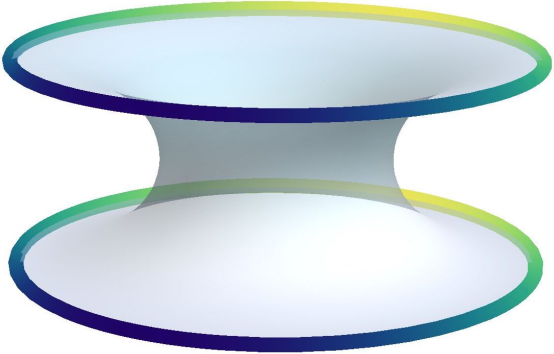
- Local solutions: Björling's fomula,...

Axially Symmetric

Let $X : \Sigma \to \mathbb{R}^3$ be a CMC $H = -c_o$ equilibria for E with $b \neq 0$. If any boundary component is a circle, then the surface is axially symmetric, i.e. a part of a Delaunay surface.

Nodoidal Domains



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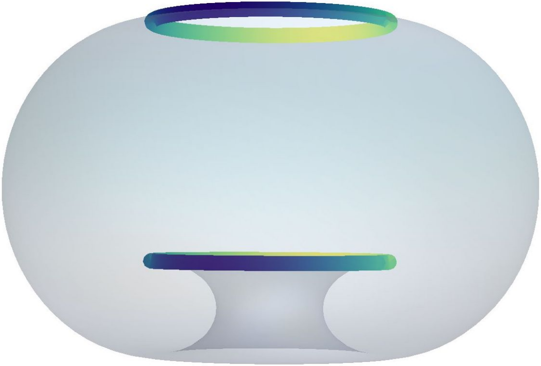

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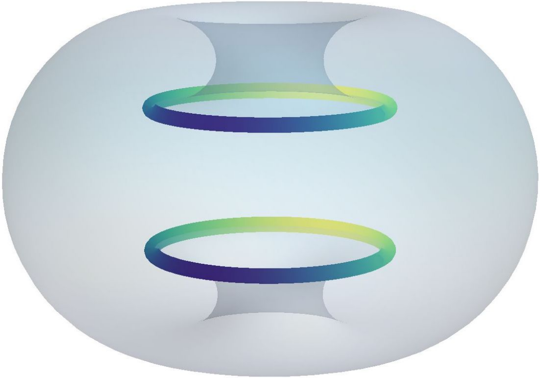

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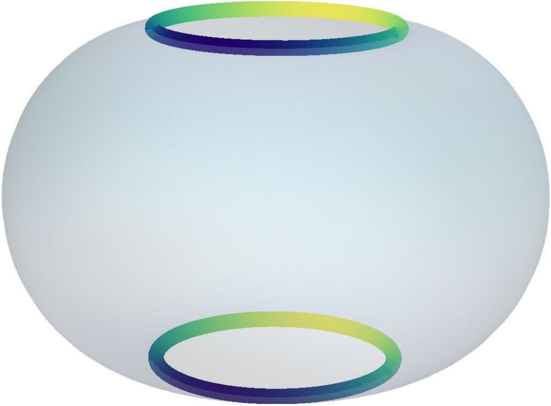

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- B. Palmer and A. Pámpano, Minimizing Configurations for Elastic Surface Energies with Elastic Boundaries, *Journal of Nonlinear Science*, **31-23** (2021).

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Thank You!