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Solutions of the Ermakov-Milne-Pinney Equation and Invariant CMC Surfaces

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Here we will consider the EMP equation with constant coefficients, i. e.

$$\alpha(s) = \alpha_o \in \mathbb{R}.$$

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where $e \in \mathbb{R}$ is only related with h and ε_i.

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Extended Blaschke's Curvature Energy

In $M_r^3(\rho)$ we are going to consider the curvature energy functional

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Euler-Lagrange equations

$$\begin{aligned} \frac{d^2}{ds^2}\left(\frac{\varepsilon_1\varepsilon_2}{\sqrt{\kappa-\mu}}\right) + \frac{1}{\sqrt{\kappa-\mu}}\left(\kappa^2 - \varepsilon_1\varepsilon_3\tau^2 + \varepsilon_2\rho\right) &= 2\kappa\sqrt{\kappa-\mu}\,, \\ \frac{d}{ds}\left(\frac{\tau}{\kappa-\mu}\right) &= 0\,. \end{aligned}$$

Binormal Evolution Surfaces

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Since $M_r^3(\rho)$ is complete,

1. Consider the one-parameter group of isometries determined by the flow of

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4. Since $\mu \in \mathbb{R}$ is fixed, S_γ has constant mean curvature.

Illustration in $\mathbb{S}^3(\rho)$

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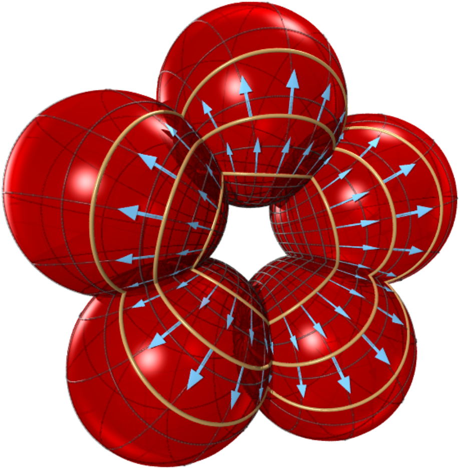

(Arroyo, Garay & –, 2019)

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THEOREM (Arroyo, Garay & –, 2018)

Let S^2 be an invariant CMC surface of $M_r^3(\rho)$ (S^2 is a warped product surface) , then the warping function is a solution of the EMP equation with constant coefficients.

References

1. J. Arroyo, O. J. Garay and A. Pámpano, Constant Mean Curvature Invariant Surfaces and Extremals of Curvature Energies, *J. Math. Anal. App.*, **462** (2018), 1644-1668.
2. J. Arroyo, O. J. Garay and A. Pámpano, Delaunay Surfaces in $\mathbb{S}^3(\rho)$, *To appear in Filomat*, (2019).
3. W. Blaschke, Vorlesungen uber Differentialgeometrie und Geometrische Grundlagen von Einsteins Relativitatstheorie I: Elementare Differenntialgeometrie, *Springer*, (1930).
4. L. Euler, Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes, Sive Solutio Problematis Isoperimetrici Lattisimo Sensu Accepti, *Bousquet, Lausannae et Genevae*, **24** (1744).
5. E. Pinney, The Nonlinear Differential Equation $y'' + p(x)y' + cy^3 = 0$, *Proc. A. M. S.*, **1** (1950).

THE END

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