Construction of Closed Biconservative Surfaces

Álvaro Pámpano Llarena

SING Seminar
"Al.I.Cuza" University of Iasi

Iasi-Virtual, February 3 (2021)

Biconservative Surfaces

Let M^{n-1} be a hypersurface in a Riemannian manifold N^n.

Biconservative Surfaces

Let M^{n-1} be a hypersurface in a Riemannian manifold N^n.

Definition

We say that M^{n-1} is biconservative if

$$2S_\eta\,(\operatorname{grad} H) + (n-1)H \operatorname{grad} H - 2H\,\operatorname{Ricci}(\eta)^T = 0$$

holds.

Biconservative Surfaces

Let M^{n-1} be a hypersurface in a Riemannian manifold N^n.

Definition

We say that M^{n-1} is biconservative if

$$2S_\eta(\operatorname{grad} H) + (n-1)H \operatorname{grad} H - 2H\,\operatorname{Ricci}(\eta)^T = 0$$

holds.

- If N^n is a space form, $N^n(\rho)$, the last term vanishes.

Biconservative Surfaces

Let M^{n-1} be a hypersurface in a Riemannian manifold N^n.

Definition

We say that M^{n-1} is biconservative if

$$2S_\eta(\operatorname{grad} H) + (n-1)H \operatorname{grad} H - 2H \operatorname{Ricci}(\eta)^T = 0$$

holds.

- If N^n is a space form, $N^n(\rho)$, the last term vanishes.
- First examples: constant mean curvature hypersurface.

Biconservative Surfaces

Let M^{n-1} be a hypersurface in a Riemannian manifold N^n.

Definition

We say that M^{n-1} is biconservative if

$$2S_\eta(\operatorname{grad} H) + (n-1)H \operatorname{grad} H - 2H \operatorname{Ricci}(\eta)^T = 0$$

holds.

- If N^n is a space form, $N^n(\rho)$, the last term vanishes.
- First examples: constant mean curvature hypersurface.

From now on we will look for proper (non-CMC) biconservative surfaces in $N^3(\rho)$.

Biconservative Surfaces in $N^3(\rho)$

Let S be a (proper) biconservative surface in $N^3(\rho)$.

Biconservative Surfaces in $N^3(\rho)$

Let S be a (proper) biconservative surface in $N^3(\rho)$.

Theorem (Cadeo, Montaldo, Oniciuc & Piu, 2014)

Proper biconservative surfaces of $N^3(\rho)$ are rotational surfaces.

Biconservative Surfaces in $N^3(\rho)$

Let S be a (proper) biconservative surface in $N^3(\rho)$.

Theorem **(Cadeo, Montaldo, Oniciuc & Piu, 2014)**

Proper biconservative surfaces of $N^3(\rho)$ are rotational surfaces. Moreover,

$$K = -3H^2 + \rho$$

holds.

Biconservative Surfaces in $N^3(\rho)$

Let S be a (proper) biconservative surface in $N^3(\rho)$.

Theorem **(Cadeo, Montaldo, Oniciuc & Piu, 2014)**

Proper biconservative surfaces of $N^3(\rho)$ are rotational surfaces.
Moreover,

$$K = -3H^2 + \rho$$

holds.

- They are Weingarten surfaces ($\mathcal{W}(H, K) = 0$).

Biconservative Surfaces in $N^3(\rho)$

Let S be a (proper) biconservative surface in $N^3(\rho)$.

Theorem **(Cadeo, Montaldo, Oniciuc & Piu, 2014)**

Proper biconservative surfaces of $N^3(\rho)$ are rotational surfaces. Moreover,

$$K = -3H^2 + \rho$$

holds.

- They are Weingarten surfaces ($\mathcal{W}(H, K) = 0$).
- They are linear Weingarten surfaces, i.e. **(Fu & Li, 2013)**

$$3\kappa_1 + \kappa_2 = 0\,,$$

where $\kappa_1 = -\kappa$.

Rotational Linear Weingarten Surfaces

Rotational Linear Weingarten Surfaces

Theorem (**López & —, 2020**)

Let $S \subset \mathbb{R}^3$ be a rotational surface satisfying

$$\kappa_1 = a\kappa_2 + b\,,$$

for $a \neq 1$ and $b \in \mathbb{R}$.

Rotational Linear Weingarten Surfaces

Theorem (López & —, 2020)

Let $S \subset \mathbb{R}^3$ be a rotational surface satisfying

$$\kappa_1 = a\kappa_2 + b\,,$$

for $a \neq 1$ and $b \in \mathbb{R}$. If γ is a profile curve of S, then the curvature κ of γ satisfies the Euler-Lagrange equation associated to the curvature energy

$$\mathbf{\Theta}_\mu(\gamma) = \int_\gamma (\kappa - \mu)^n$$

where $\mu = -b/(a-1)$ and $n = a/(a-1)$.

Rotational Linear Weingarten Surfaces

Theorem (López & —, 2020)

Let $S \subset \mathbb{R}^3$ be a rotational surface satisfying

$$\kappa_1 = a\kappa_2 + b\,,$$

for $a \neq 1$ and $b \in \mathbb{R}$. If γ is a profile curve of S, then the curvature κ of γ satisfies the Euler-Lagrange equation associated to the curvature energy

$$\mathbf{\Theta}_\mu(\gamma) = \int_\gamma (\kappa - \mu)^n$$

where $\mu = -b/(a-1)$ and $n = a/(a-1)$.

- Biconservative case: $\mu = 0$ and $n = 1/4$.

Curvature Energy Functional

Curvature Energy Functional

We consider the curvature energy functional

$$\mathbf{\Theta}(\gamma) := \int_\gamma \kappa^{1/4} = \int_0^L \kappa^{1/4}(s)ds = \int_0^1 \kappa^{1/4}(t)v(t)\, dt$$

acting on the space of smooth immersed curves in Riemannian 2-space forms $N^2(\rho)$, i.e. $\gamma : [0, L] \to N^2(\rho)$.

Curvature Energy Functional

We consider the curvature energy functional

$$\mathbf{\Theta}(\gamma) := \int_\gamma \kappa^{1/4} = \int_0^L \kappa^{1/4}(s)ds = \int_0^1 \kappa^{1/4}(t)v(t)\,dt$$

acting on the space of smooth immersed curves in Riemannian 2-space forms $N^2(\rho)$, i.e. $\gamma : [0, L] \to N^2(\rho)$.

Euler-Lagrange equation

Regardless of the boundary conditions, any critical curve for $\mathbf{\Theta}$ must satisfy

$$\kappa^{3/4}\frac{d^2}{ds^2}\left(\frac{1}{\kappa^{3/4}}\right) - 3\kappa^2 + \rho = 0\,.$$

We will call them, simply, critical curves.

Killing Vector Fields Along Curves

Killing Vector Fields Along Curves

A vector field W along γ, is said to be a Killing vector field along γ if the following equations hold

$$W(v) = W(\kappa) = 0$$

along γ. **(Langer & Singer, 1984)**

Killing Vector Fields Along Curves

A vector field W along γ, is said to be a Killing vector field along γ if the following equations hold

$$W(v) = W(\kappa) = 0$$

along γ. **(Langer & Singer, 1984)**

Proposition **(Langer & Singer, 1984)**

Consider $N^2(\rho)$ embedded as a totally geodesic surface of $N^3(\rho)$. Then, the vector fields

$$\mathcal{I} = \frac{1}{4\kappa^{3/4}} B \,,$$
$$\mathcal{J} = -\frac{3}{4}\kappa^{1/4} T + \frac{d}{ds}\left(\frac{1}{4\kappa^{3/4}}\right) N$$

are Killing vector fields along critical curves.

Binormal Evolution Surfaces

Binormal Evolution Surfaces

Let $\gamma(s) \subset N^2(\rho)$ be any critical curve for $\mathbf{\Theta}$. (We consider $N^2(\rho) \subset N^3(\rho)$ and γ being planar, i.e. $\tau = 0$.)

Binormal Evolution Surfaces

Let $\gamma(s) \subset N^2(\rho)$ be any critical curve for $\mathbf{\Theta}$. (We consider $N^2(\rho) \subset N^3(\rho)$ and γ being planar, i.e. $\tau = 0$.)

1. Consider the Killing vector field along γ in the direction of the (constant) binormal vector field:

$$\mathcal{I} = \frac{1}{4\kappa^{3/4}} B \,.$$

Binormal Evolution Surfaces

Let $\gamma(s) \subset N^2(\rho)$ be any critical curve for $\mathbf{\Theta}$. (We consider $N^2(\rho) \subset N^3(\rho)$ and γ being planar, i.e. $\tau = 0$.)

1. Consider the Killing vector field along γ in the direction of the (constant) binormal vector field:

$$\mathcal{I} = \frac{1}{4\kappa^{3/4}} B \, .$$

2. Let's denote by ξ the (unique) extension to a Killing vector field of $N^3(\rho)$. (It can be assumed to be: $\xi = \lambda_1 X_1 + \lambda_2 X_2$.)

Binormal Evolution Surfaces

Let $\gamma(s) \subset N^2(\rho)$ be any critical curve for $\mathbf{\Theta}$. (We consider $N^2(\rho) \subset N^3(\rho)$ and γ being planar, i.e. $\tau = 0$.)

1. Consider the Killing vector field along γ in the direction of the (constant) binormal vector field:

$$\mathcal{I} = \frac{1}{4\kappa^{3/4}} B \, .$$

2. Let's denote by ξ the (unique) extension to a Killing vector field of $N^3(\rho)$. (It can be assumed to be: $\xi = \lambda_1 X_1 + \lambda_2 X_2$.)
3. Since $N^3(\rho)$ is complete, the one-parameter group of isometries determined by ξ is $\{\phi_t \, , \, t \in \mathbb{R}\}$.

Binormal Evolution Surfaces

Let $\gamma(s) \subset N^2(\rho)$ be any critical curve for $\mathbf{\Theta}$. (We consider $N^2(\rho) \subset N^3(\rho)$ and γ being planar, i.e. $\tau = 0$.)

1. Consider the Killing vector field along γ in the direction of the (constant) binormal vector field:

$$\mathcal{I} = \frac{1}{4\kappa^{3/4}} B \,.$$

2. Let's denote by ξ the (unique) extension to a Killing vector field of $N^3(\rho)$. (It can be assumed to be: $\xi = \lambda_1 X_1 + \lambda_2 X_2$.)
3. Since $N^3(\rho)$ is complete, the one-parameter group of isometries determined by ξ is $\{\phi_t \,,\, t \in \mathbb{R}\}$.
4. We construct the binormal evolution surface **(Garay & —, 2016)**

$$S_\gamma := \{x(s,t) := \phi_t\left(\gamma(s)\right)\} \,.$$

Geometric Properties

Geometric Properties

By construction S_γ is a ξ-invariant surface.

Geometric Properties

By construction S_γ is a ξ-invariant surface. Moreover, it verifies:

- Since $\gamma(s) \subset N^2(\rho)$ (γ is planar),

Theorem **(Arroyo, Garay & —, 2017)**

The binormal evolution surface S_γ is either a flat isoparametric surface (when $\kappa(s) = \kappa_o$ is constant);

Geometric Properties

By construction S_γ is a ξ-invariant surface. Moreover, it verifies:

- Since $\gamma(s) \subset N^2(\rho)$ (γ is planar),

Theorem (Arroyo, Garay & —, 2017)

The binormal evolution surface S_γ is either a flat isoparametric surface (when $\kappa(s) = \kappa_o$ is constant); or, it is a rotational surface (when $\kappa(s)$ is not constant).

Geometric Properties

By construction S_γ is a ξ-invariant surface. Moreover, it verifies:

- Since $\gamma(s) \subset N^2(\rho)$ (γ is planar),

Theorem (Arroyo, Garay & —, 2017)

The binormal evolution surface S_γ is either a flat isoparametric surface (when $\kappa(s) = \kappa_o$ is constant); or, it is a rotational surface (when $\kappa(s)$ is not constant). In particular, spherical rotational surface if $d > 0$ holds (constant of integration).

Geometric Properties

By construction S_γ is a ξ-invariant surface. Moreover, it verifies:

- Since $\gamma(s) \subset N^2(\rho)$ (γ is planar),

Theorem (Arroyo, Garay & —, 2017)

The binormal evolution surface S_γ is either a flat isoparametric surface (when $\kappa(s) = \kappa_o$ is constant); or, it is a rotational surface (when $\kappa(s)$ is not constant). In particular, spherical rotational surface if $d > 0$ holds (constant of integration).

- Since $\gamma(s)$ is a critical curve for $\mathbf{\Theta}$,

Theorem (Montaldo & —, 2020)

The binormal evolution surface S_γ is a proper biconservative surface. It verifies:

$$3\kappa_1 + \kappa_2 = 0 \,.$$

Closure Conditions

Searching for closed (proper) biconservative surfaces, we need:

Closure Conditions

Searching for closed (proper) biconservative surfaces, we need:

- Spherical rotation, i.e. $d > 0$.

Closure Conditions

Searching for closed (proper) biconservative surfaces, we need:

- Spherical rotation, i.e. $d > 0$.
- Closed profile curve, i.e. closed critical curve for $\mathbf{\Theta}$.

Closure Conditions

Searching for closed (proper) biconservative surfaces, we need:

- Spherical rotation, i.e. $d > 0$.
- Closed profile curve, i.e. closed critical curve for $\mathbf{\Theta}$.

Closure Conditions

Let $\gamma(s) \subset N^2(\rho)$ be a critical curve for $\mathbf{\Theta}$ with periodic curvature. Then, $\gamma(s)$ is closed if and only if

$$\Lambda(d) = 12 \int_0^{\varrho} \frac{\kappa^{7/4}}{16 d \kappa^{3/2} - \rho} \, ds$$

equals 0 for $\rho \leq 0$

Closure Conditions

Searching for closed (proper) biconservative surfaces, we need:

- Spherical rotation, i.e. $d > 0$.
- Closed profile curve, i.e. closed critical curve for $\mathbf{\Theta}$.

Closure Conditions

Let $\gamma(s) \subset N^2(\rho)$ be a critical curve for $\mathbf{\Theta}$ with periodic curvature. Then, $\gamma(s)$ is closed if and only if

$$\Lambda(d) = 12 \int_0^{\varrho} \frac{\kappa^{7/4}}{16 d \kappa^{3/2} - \rho} \, ds$$

equals 0 for $\rho \leq 0$, or $2n\pi/(m\sqrt{\rho d})$ for $\rho > 0$.

Existence of Closed Biconservative Surfaces

Proposition (Montaldo & —, 2020)

There are not closed non-CMC biconservative surfaces in $N^3(\rho)$ with $\rho \leq 0$.

Existence of Closed Biconservative Surfaces

Proposition (Montaldo & —, 2020)

There are not closed non-CMC biconservative surfaces in $N^3(\rho)$ with $\rho \leq 0$.

- First obtained in **(Nistor & Oniciuc, 2019-2020)**, using a different technique.

Existence of Closed Biconservative Surfaces

Proposition (Montaldo & —, 2020)

There are not closed non-CMC biconservative surfaces in $N^3(\rho)$ with $\rho \leq 0$.

- First obtained in **(Nistor & Oniciuc, 2019-2020)**, using a different technique.
- However, we will prove the existence in $\mathbb{S}^3(\rho)$.

Existence of Closed Biconservative Surfaces

Proposition (Montaldo & —, 2020)

There are not closed non-CMC biconservative surfaces in $N^3(\rho)$ with $\rho \leq 0$.

- First obtained in **(Nistor & Oniciuc, 2019-2020)**, using a different technique.
- However, we will prove the existence in $\mathbb{S}^3(\rho)$.

Proposition (Montaldo & —, 2020)

Critical curves for $\mathbf{\Theta}$ in $\mathbb{S}^2(\rho)$ have periodic curvature.

Idea of the Proof

1. Let $x = \kappa^{1/2}$ and $y = x'$, then the first integral of the Euler-Lagrange equation reads

$$y^2 = \frac{4}{9}x^2\left(16dx^3 - 9x^4 - \rho\right) = \frac{4}{9}x^2 Q(x) \, .$$

Idea of the Proof

1. Let $x = \kappa^{1/2}$ and $y = x'$, then the first integral of the Euler-Lagrange equation reads

$$y^2 = \frac{4}{9}x^2\left(16dx^3 - 9x^4 - \rho\right) = \frac{4}{9}x^2 Q(x)\,.$$

2. The constant of integration: $d > d_* = (27\rho)^{1/4}/4 > 0$.

Idea of the Proof

1. Let $x = \kappa^{1/2}$ and $y = x'$, then the first integral of the Euler-Lagrange equation reads

$$y^2 = \frac{4}{9}x^2\left(16dx^3 - 9x^4 - \rho\right) = \frac{4}{9}x^2 Q(x)\,.$$

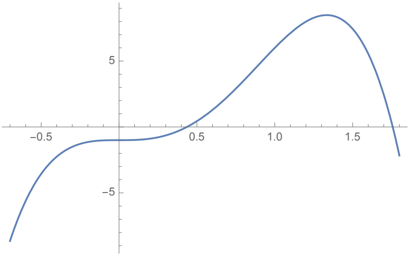
2. The constant of integration: $d > d_* = (27\rho)^{1/4}/4 > 0$.
3. Square root method and Poincare-Bendixon Theorem.

Idea of the Proof

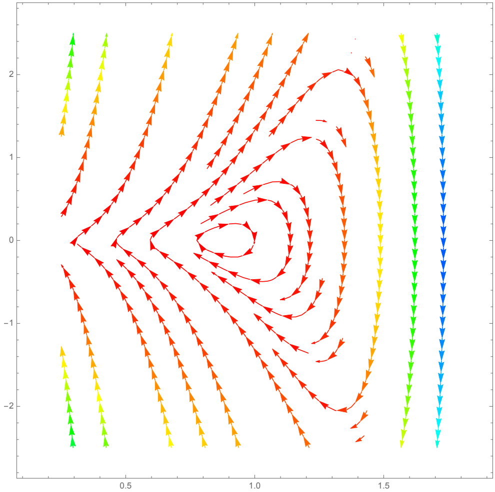
1. Let $x = \kappa^{1/2}$ and $y = x'$, then the first integral of the Euler-Lagrange equation reads

$$y^2 = \frac{4}{9}x^2\left(16dx^3 - 9x^4 - \rho\right) = \frac{4}{9}x^2 Q(x)\,.$$

2. The constant of integration: $d > d_* = (27\rho)^{1/4}/4 > 0$.
3. Square root method and Poincare-Bendixon Theorem.



Idea of the Proof



Existence of Closed Critical Curves

Lemma (Montaldo & —, 2020)

The function $I(d) = \sqrt{\rho\, d}\,\Lambda(d)$ decreases in $d \in (d_*, \infty)$. Moreover,

$$\sqrt{2}\,\pi > I(d) > \pi\,.$$

Existence of Closed Critical Curves

Lemma (Montaldo & —, 2020)

The function $I(d) = \sqrt{\rho\, d}\,\Lambda(d)$ decreases in $d \in (d_*, \infty)$.
Moreover,

$$\sqrt{2}\,\pi > I(d) > \pi\,.$$

Theorem (Montaldo & —, 2020)

There exists a discrete biparametric family of closed non-CMC biconservative surfaces in $\mathbb{S}^3(\rho)$. None of them is embedded.

Existence of Closed Critical Curves

Lemma (Montaldo & —, 2020)

The function $I(d) = \sqrt{\rho\, d}\,\Lambda(d)$ decreases in $d \in (d_*, \infty)$.
Moreover,

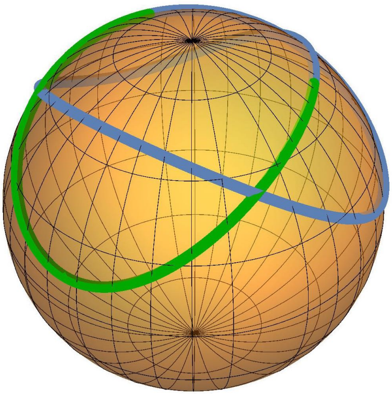
$$\sqrt{2}\,\pi > I(d) > \pi\,.$$

Theorem (Montaldo & —, 2020)

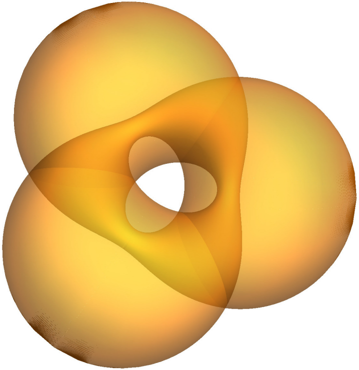
There exists a discrete biparametric family of closed non-CMC biconservative surfaces in $\mathbb{S}^3(\rho)$. None of them is embedded.

- For any m and n such that, $m < 2n < \sqrt{2}m$, we have a closed non-CMC biconservative surface.

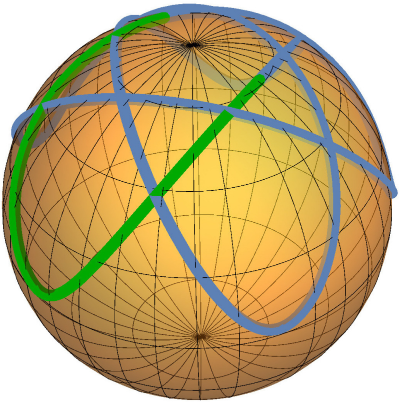
Critical Curve for $\boldsymbol{\Theta}$ ($m = 3$ and $n = 2$)



Closed Biconservative Surface ($m = 3$ and $n = 2$)



Critical Curve for $\mathbf{\Theta}$ ($m = 5$ and $n = 3$)



Proof of Lemma

Lemma (Montaldo & —, 2020)

The function $I(d) = \sqrt{\rho\, d}\, \Lambda(d)$ decreases in $d \in (d_*, \infty)$.
Moreover,

$$\sqrt{2}\,\pi > I(d) > \pi\,.$$

Proof of Lemma

Lemma (Montaldo & —, 2020)

The function $I(d)=\sqrt{\rho\, d}\,\Lambda(d)$ decreases in $d\in(d_*,\infty)$. Moreover,

$$\sqrt{2}\,\pi > I(d) > \pi\,.$$

- When $d\to d_*$, we use a Dirac's Delta. Also, a result of **(Perdomo, 2010)**.

Proof of Lemma

Lemma (Montaldo & —, 2020)

The function $I(d) = \sqrt{\rho\, d}\, \Lambda(d)$ decreases in $d \in (d_*, \infty)$. Moreover,

$$\sqrt{2}\,\pi > I(d) > \pi\,.$$

- When $d \to d_*$, we use a Dirac's Delta. Also, a result of **(Perdomo, 2010)**.
- When $d \to \infty$, we use Complex Analysis (Cauchy's Integral Formula and Residues).

Proof of Lemma

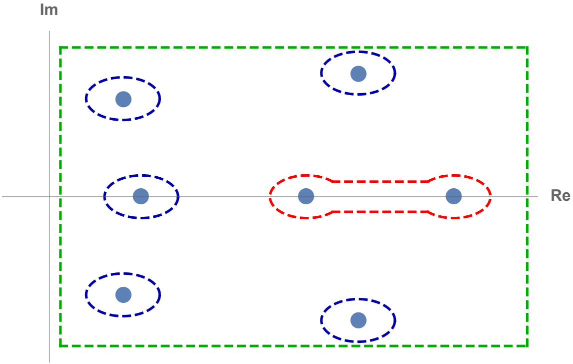
Lemma (Montaldo & —, 2020)

The function $I(d) = \sqrt{\rho\, d}\, \Lambda(d)$ decreases in $d \in (d_*, \infty)$. Moreover,

$$\sqrt{2}\,\pi > I(d) > \pi\,.$$

- When $d \to d_*$, we use a Dirac's Delta. Also, a result of **(Perdomo, 2010)**.
- When $d \to \infty$, we use Complex Analysis (Cauchy's Integral Formula and Residues).
- We also use Complex Analysis to compute the first derivative.

Proof of Lemma



THE END

Thank You!