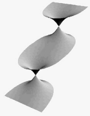

Constant Mean Curvature Invariant Surfaces in $\mathbb{L}^3$ and a Blaschke's Variational Problem

Álvaro Pámpano Llarena

Young Researcher Workshop on Differential Geometry in Minkowski Space

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Characterize constant mean curvature ξ-invariant surfaces S of $\mathbb{L}^3$.

For this purpose,

- Get a geodesic foliation of S.
- Leaves are critical for the Blaschke's problem.
- Finally, study isometric deformations.

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2. Reduction Theorem

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5. Planar Critical Curves

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- Take into account that $\kappa = \mu$ would be a global minimum if we were considering $L^1([0, L])$ as the space of curves.
- Observe that the case $\mu = 0$ was studied by Blaschke in the Euclidean 3-Space, [2].

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Reduction Theorem ([1])

A critical point of Θ must lie in a 3-dimensional totally geodesic submanifold of $\mathbb{L}^n$.

Thus, we are interested in studying critical curves in the Minkowski 3-Space, $\mathbb{L}^3$.

Euler-Lagrange Equations

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The Euler-Lagrange equations for the curvature energy functional $\Theta(\gamma) = \int_\gamma \sqrt{\kappa - \mu}$, in $\mathbb{L}^3$ can be written as

$$\begin{aligned} \frac{d^2}{ds^2}\left(\frac{\varepsilon_2}{\sqrt{\kappa-\mu}}\right) + \frac{1}{\sqrt{\kappa-\mu}}(\varepsilon_1\kappa^2 - \varepsilon_3\tau^2) &= 2\varepsilon_1\kappa\sqrt{\kappa-\mu}\,, \\ \frac{d}{ds}\left(\frac{\tau}{\kappa-\mu}\right) &= 0\,. \end{aligned}$$

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Under suitable boundary conditions, solutions of these equations are critical curves for our energy functional.

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- Thus, γ must be a Frenet helix.
- Moreover, substituting this in the first Euler-Lagrange equation we get the relation

$$\kappa_o = \mu + \sqrt{\mu^2 - \varepsilon_1 \varepsilon_3 \tau_o^2}.$$

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$$\begin{aligned} a &= -\varepsilon_1\varepsilon_2\mu^2\,, \\ b &= 4\varepsilon_2 d + 2\varepsilon_1\varepsilon_2\mu\,, \\ c &= -\varepsilon_1\varepsilon_2 - \varepsilon_2\varepsilon_3 e^2\,, \end{aligned}$$

and $\Delta = 4ac - b^2 = -16d^2 - 16\varepsilon_1\mu d + 4\varepsilon_1\varepsilon_3\mu^2 e^2$, where d, e are real constants (constants of integration).

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Thus, the following cases are not possible:

1. $\Delta \geq 0$ and $c < 0$,
2. $a \leq 0$, $2d = -\varepsilon_1\mu$ and $e^2 = -\varepsilon_1\varepsilon_3$.

Solutions with Non-Constant Curvature

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1. If $\Delta \neq 0$ and $a \neq 0$

$$\kappa(s) = \frac{2a + \mu(b + \sqrt{|\Delta|} f(2\mu s))}{-b + \sqrt{|\Delta|} f(2\mu s)} ,$$

where, $f(x) = \sinh x$, if $\Delta > 0$ and $a > 0$; $f(x) = \cosh x$, if $\Delta < 0$ and $a > 0$; and $f(x) = \sin x$, if $\Delta < 0$ and $a < 0$.

Solutions with Non-Constant Curvature

2. If $\Delta = 0$ and $a > 0$

$$k(s) = \frac{\mu + (2a - b\mu)\exp 2\mu s}{1 - b\exp 2\mu s}\,.$$

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4. If $\Delta = 0$ and $a = 0$, that is, $(\mu = d = 0)$

$$k(s) = \frac{1}{2\sqrt{cs}} .$$

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The curvature and torsion (together with the causal characters ε_i of the Frenet frame) determine a unique curve up to rigid motions.

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The curvature and torsion (together with the causal characters ε_i of the Frenet frame) determine a unique curve up to rigid motions.

- When $\mu = 0$, our critical curves are Lancret curves, that is, curves making a constant angle with a fixed direction.
- On the other hand, if $\mu \neq 0$, our critical curves are Bertrand curves.

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Delaunay Curves ([1], [3])

Critical curves of $\mathbf{\Theta}(\gamma) = \int_\gamma \sqrt{\kappa - \mu}\, ds$ in $\mathbb{R}^2$ are precisely the Delaunay curves, that is, the roulettes of foci of conics (lines, circles, catenaries, nodaries and undularies).

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If γ lies in a Lorentzian plane, $\gamma \subset \mathbb{L}^2$,we can prove

Hano-Nomizu Curves ([1], [5])

The locus of the origin when a part of a spacelike quadratic curve is rolled along a spacelike line is a spacelike critical curve for $\mathbf{\Theta}(\gamma) = \int_\gamma \sqrt{\kappa - \mu}\, ds$ in $\mathbb{L}^2$.

CMC Invariant Surfaces

1. Killing Vector Fields

CMC Invariant Surfaces

1. Killing Vector Fields
2. Evolution of Critical Curves

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3. Geodesic Foliations of Invariant Surfaces

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Associated Killing Vector Field along γ ([3])

The vector field $\mathcal{I} = \frac{1}{2\sqrt{\kappa-\mu}} B$ is a Killing vector field along γ, if and only if, γ verifies the Euler-Lagrange equations (γ may have a restriction on its length).

Evolution of Critical Curves

Since $\mathbb{L}^3$ is complete,

1. Consider the one-parameter group of isometries determined by the flow of
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Foliations of Invariant Surfaces

For the converse, assume that S is a non-degenerate $\mathcal{G}_\xi$-invariant surface of $\mathbb{L}^3$, i.e., for any $x \in S$ and $\Phi_t \in \mathcal{G}_\xi$ we have $\Phi_t(S) = S$.

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Geodesic Foliation by Critical Curves ([1])

A ξ-invariant CMC surface S of $\mathbb{L}^3$ admits a local geodesic parametrization where the leaves provide a geodesic foliation by critical curves of the extended Blaschke's problem with $\mu = -\varepsilon_1\varepsilon_2 H$.

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Thus, every ξ-invariant CMC surface is

- A ruled surface or,
- It is generated by evolving a critical curve of $\boldsymbol{\Theta}(\gamma) = \int_\gamma \sqrt{\kappa + \varepsilon_1\varepsilon_2 H}\, ds$ under the flow of the Killing vector field ξ.

Isometric Deformations

1. Deformations by Isometric Surfaces

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2. Orbits of Deformations

Deformations by Isometric Surfaces (1)

Suppose that the critical curve γ of $\boldsymbol{\Theta}(\gamma) = \int_\gamma \sqrt{\kappa - \mu}\, ds$ has constant curvature κ_o

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Suppose that the critical curve γ of $\mathbf{\Theta}(\gamma) = \int_\gamma \sqrt{\kappa - \mu}\, ds$ has constant curvature κ_o, then we know that γ is a Frenet helix with

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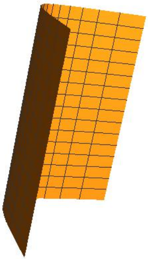
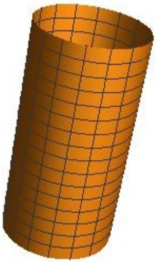
$$\begin{aligned} \kappa &= \kappa_o = \mu + \sqrt{\mu^2 - \varepsilon_2\varepsilon_3\tau_o^2}\,, \\ \tau &= \tau_o\,, \end{aligned}$$

and these helices can only generate congruence surfaces to the following ones (depending on the causal character of ε_i)



(E) $\varepsilon_1\varepsilon_2 = -1$



(F) $\varepsilon_3 = -1$

Deformations by Isometric Surfaces (2)

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If the following relations involving the constants d and e,

1. If $\Delta \neq 0$ and $a \neq 0$, $\Delta = \nu b^2$,
2. If $\Delta = 0$ and $a > 0$, (this does not give an isometric deformation),
3. If $\Delta < 0$ and $a = 0$, $c = \nu b^2$ and
4. If $\Delta = a = 0$, (as $d = 0$, there is no biparametric family) there is no any constraint,

are verified for some $\nu \in \mathbb{R}$ and for each correpondent solution of the Euler-Lagrange equations

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are verified for some $\nu \in \mathbb{R}$ and for each correpondent solution of the Euler-Lagrange equations, we can prove

Isometric Deformations ([1])

For each real constant μ, let $\{S_\gamma\}_{d,e}$ be the family of ξ-invariant surfaces shaped on a critical curve γ of
$\mathbf{\Theta}(\gamma) = \int_\gamma \sqrt{\kappa - \mu}\, ds$.

Deformations by Isometric Surfaces (2)

If the following relations involving the constants d and e,

1. If $\Delta \neq 0$ and $a \neq 0$, $\Delta = \nu b^2$,
2. If $\Delta = 0$ and $a > 0$, (this does not give an isometric deformation),
3. If $\Delta < 0$ and $a = 0$, $c = \nu b^2$ and
4. If $\Delta = a = 0$, (as $d = 0$, there is no biparametric family) there is no any constraint,

are verified for some $\nu \in \mathbb{R}$ and for each correpondent solution of the Euler-Lagrange equations, we can prove

Isometric Deformations ([1])

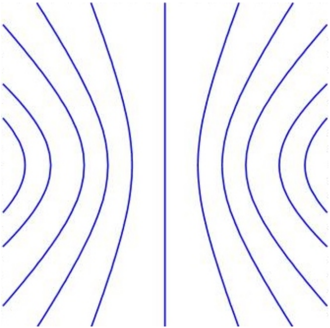
For each real constant μ, let $\{S_\gamma\}_{d,e}$ be the family of ξ-invariant surfaces shaped on a critical curve γ of $\mathbf{\Theta}(\gamma) = \int_\gamma \sqrt{\kappa - \mu}\, ds$.Under the relations above (except for case 2), the family $\{S_\gamma\}_d$ is generated by isometric surfaces with the same constant mean curvature $H = -\varepsilon_1 \varepsilon_2 \mu$.

Deformations of Riemannian CMC Surfaces

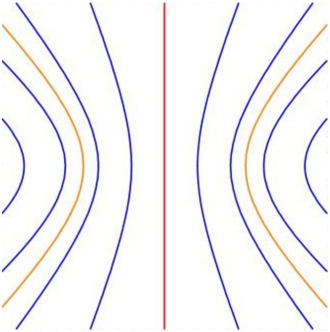
Let S_γ be a Riemannian surface of $\mathbb{L}^3$,

Deformations of Riemannian CMC Surfaces

Let S_γ be a Riemannian surface of $\mathbb{L}^3$,then the orbits are



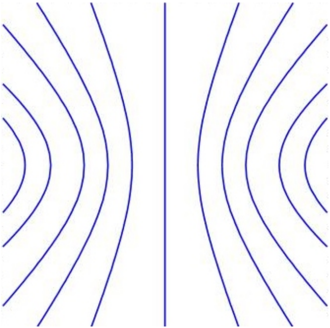
(I) $H = 0$



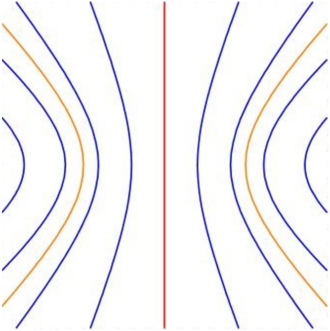
(J) $H \neq 0$

Deformations of Riemannian CMC Surfaces

Let S_γ be a Riemannian surface of $\mathbb{L}^3$,then the orbits are



(K) $H = 0$



(L) $H \neq 0$

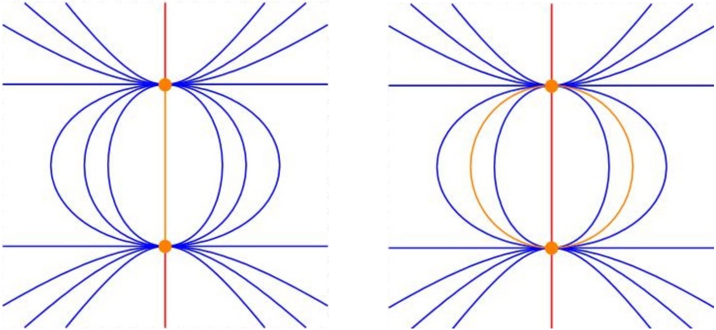
Rotational Spacelike Surfaces ([1])

Any spacelike surface of CMC can be isometrically deformed into a spacelike rotational CMC surface, except for the yellow cases.

Deformations of Lorentzian Surfaces

Let S_γ be a Lorentzian surface of $\mathbb{L}^3$ with timelike profile curve γ.

Let S_γ be a Lorentzian surface of $\mathbb{L}^3$ with timelike profile curve γ. Then, we have the following orbits of the isometric deformations

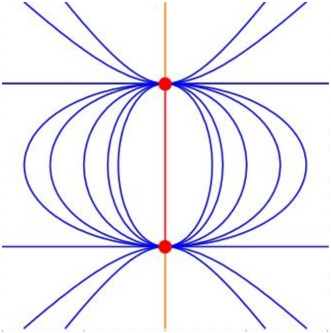


(o) $H = 0$ (p) $H \neq 0$

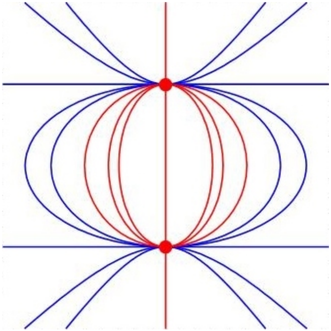
Now, if γ is a spacelike profile curve of a Lorentzian surface S_γ of $\mathbb{L}^3$

Deformations of Lorentzian Surfaces

Now, if γ is a spacelike profile curve of a Lorentzian surface S_γ of $\mathbb{L}^3$, the isometric deformations appear in the following diagrams



(S) $H = 0$



(T) $H \neq 0$

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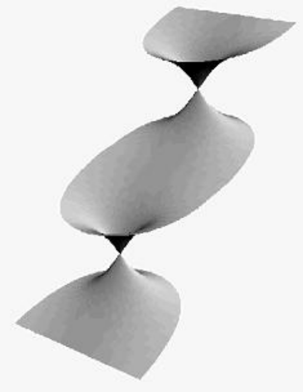
THE END



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