A note on p-elasticae and the generalized EMP equation

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A correspondence is shown to exist between *p-elastic curves* (a generalization of the classical Euler elastica) of the Euclidean and Minkowski 3-spaces and a certain family of generalized Ermakov-Milne-Pinney (EMP) equations. Actually, combining the Euler-Lagrange equations of the p-elastica variational problem with the Frenet formulae for regular curves, a relation between the p-elastica curvature and some generalized EMP equations is obtained. As a consequence, since p-elasticae are totally determined by their curvature, they can be seen as geometric solutions of this class of generalized EMP equations and can be used as an alternative route for obtaining exact solutions of them, as illustrated by a couple of examples. As another consequence, first integrals of these nonlinear equations are derived.

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1. Introduction

Nonlinear equations have been of an increasing interest in Physics during the last decades and one of the simplest examples is the today called *Ermakov-Milne-Pinney (EMP) equation*, i.e., the second order nonlinear differential equation

$$x''(t) + q(t)x(t) = \frac{h}{x^3(t)}, \tag{1.1}$$

h being a constant. If $h = 0$, this is a particular instance of the 1-dimensional linear Schrödinger equation and for $h \neq 0$ and $q(t) = \omega^2(t)$ it describes the radial equation of motion for a two-dimensional time-dependent linear oscillator, [9]. It was introduced by V.P. Ermakov as a way of looking for a first integral for the time-dependent harmonic oscillator [10] and it may be regarded as the simplest example of an Ermakov system, [20]. Recently, various authors have shown that, for a number of pure scalar field and other classes of cosmological models, Einstein's gravitational field equations can be equivalently reformulated in terms of the so called *generalized EMP* (GEMP) equation

$$x''(t) + q(t)x(t) = \sum_{i=0}^{m} \frac{\lambda_i(t)}{x(t)^{b_i}},$$

for functions $q(t), \lambda_i(t)$ and $b_i \in \mathbb{R}$, $i \in \{0, ..., m\}$ (see, for example, [7], [8], [15]).

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In this note we prove that solutions of a certain type of GEMP equations are characterized geometrically, in the sense that they are shown to be in correspondence with stationary (also called critical) curves γ for a family of functionals, acting on certain spaces of curves in the Euclidean and Minkowski 3-spaces), whose lagrangian density depends upon curvature as described by

$$\mathscr{F}_p^\mu(\gamma) = \int_\gamma (\kappa - \mu)^p \, ds, \tag{1.2}$$

κ denoting the curvature of the curve and $\mu, p \in \mathbb{R}$ (for more details see section 2). Lagrangian densities including also the torsion of the curve have been considered, for example, in [1, 5].

For a certain choice of parameters ($\mu = 0$, $p = 2$), the family of functionals (1.2) includes the Euler-Bernoulli *bending energy* whose minimizers are, precisely, the classical *Euler-Bernoulli elasticae* (that is why here critical curves of (1.2) are referred to as *p-elasticae*). In combination with the Palais' criticality principle and/or other reduction procedures, elasticae can be used, for example, to obtain closed form solutions for the Helfrich-Canham model for biomembranes in Biophysics [14, 18], explicit examples of worldsheets for the Kleinert-Polyakov action in string theory [4]. They can also provide invariant solutions for the vortex filament equation in Fluid Dynamics [13, 16], and soliton solutions of the nonlinear cubic Schrödinger equation [12].

Here, using a different and direct approach, what we actually establish is a correspondence between the solutions of certain GEMP equations and the p-elasticae's curvatures, but it turns out that the curvature of a p-elastica determines the torsion (see equation (3.13)) and both, curvature and torsion, completely determines in turn the curve itself (up to congruences of the background space). Hence, the p-elasticae themselves can be considered as solutions of this family of GEMP equations. Moreover, two examples are given in which this correspondence is used to provide exact solutions of the associated GEMP equations.

Finally, as noticed before, Ermakov used (1.1) to obtain a first integral for the 1-dimensional linear Schrödinger equation $x'' + w(t)^2 x = 0$. In the last section, exploiting the geometric correspondence mentioned in the above paragraph, we derive first integrals of the GEMP equations involved in this work.

2. p-elastica in the Euclidean or Minkowski 3-spaces

In $\mathbb{R}^3$ with coordinates (x, y, z) we consider the metric $g = dx^2 + dy^2 + (-1)^r dz^2$, where $r \in \{0, 1\}$. Then, $\mathbb{R}^3$ endowed with the metric g (g will usually be denoted by $\langle \, , \, \rangle$) is the standard *Euclidean space*, $\mathbb{E}^3$, when $r = 0$, while if $r = 1$, the resulting space is the *Minkowski* 3*-space*, $\mathbb{E}^3_1$.

If $\gamma : I \to \mathbb{E}^3_r$, $r \in \{0, 1\}$, is a smooth curve defined on an interval I and immersed in either of the two above flat spaces, $\gamma'(t) = \frac{d\gamma(t)}{dt}$ will represent its tangent (or *velocity*) vector, while $\|\gamma'(t)\|$ will be its *speed*. As usual, the apostrophe $()'$ will denote the derivative with respect to the curve parameter. A curve γ is said to be *lightlike* (or *null*) if $\langle \gamma'(t), \gamma'(t) \rangle = 0$, $\forall\, t \in I$. Along this paper we are going to consider only non-null curves with non-null *acceleration* $\gamma''(t)$. A non-null curve can be parametrized by the arc-length and this natural parameter, denoted by s as usual, is called *proper time*. Let $\gamma(s)$ be a unit speed non-geodesic curve with non-null velocity $\gamma'(s) = T(s)$, i.e., $\langle T, T \rangle = \pm 1 = \varepsilon_1$, and non-null acceleration $T'(s)$. Then $T' = \varepsilon_2 \kappa(s) N$, N being a unit normal field along $\gamma(s)$ and ε_2 its causal character, $\langle N, N \rangle = \pm 1 = \varepsilon_2$, defines the curvature $\kappa(s)$ of the curve. Now, the binormal B is defined by $B = \varepsilon_3 T \times N$, $\varepsilon_3 = \langle T \times N, T \times N \rangle = \pm 1$ denoting the causal character of $T\times N$. Then, it is easy to see that the following *Frenet equations* hold in $\mathbb{E}^3_r$, $r\in\{0,1\}$

$$\begin{aligned} T' &= \varepsilon_2\kappa(s)N\,,\\ N' &= -\varepsilon_1\kappa(s)T+\varepsilon_3\tau(s)B\,,\\ B' &= -\varepsilon_2\tau(s)N\,, \end{aligned} \tag{2.1}$$

and define both the *curvature*, $\kappa(s)$, and *torsion*, $\tau(s)$, of γ. Of course, if $r=0$, the *causal characters* are $\varepsilon_i=1$, $1\leq i\leq 3$.

To fix ideas, assume that $\gamma\colon[a_o,a_1]\to\mathbb{E}^3_r$, $t\to\gamma(t)$, is an immersed curve with non-null velocity and acceleration. In the next section, for given $p,\mu\in\mathbb{R}$, we are going to consider curves which are critical for energy functionals of the type

$$\mathscr{F}^\mu_p(\gamma)=\int_0^L(\kappa(s)-\mu)^p\,ds=\int_{a_o}^{a_1}(\kappa(s(t))-\mu)^p\,v(t)\,dt\,, \tag{2.2}$$

$s:[a_o,a_1]\to[0,L]$ being defined by $s(t):=\int_{a_o}^t\|\gamma'(t)\|\,dt$, and where L denotes the length of $\gamma(t)$ between $\gamma(a_o)$ and $\gamma(a_1)$ and $v(t)=\|\gamma'(t)\|$ represents the norm of the tangent vector to the curve. As before, the arc-length parameter is represented by $s\in[0,L]$. In general, $\mathscr{F}^\mu_p$ will be considered acting on spaces of curves with the same length satisfying suitable boundary conditions. Observe that if $p=0$, then (2.2) is nothing but the *length functional* whose critical curves are *geodesics*, and if $p=1$, then (2.2) is, basically, the *total curvature functional* in which case extremals are the planar curves [2]. Thus, in the next section, we are going to consider $p\in\mathbb{R}-\{0,1\}$. When $p=2$ and $\mu=0$, then (2.2) represents the *bending energy* of a rod parametrized by γ, and its minimizers (under suitable boundary conditions and constraints) correspond to the classical *elastic rods, or, simply, elasticae*, the mathematical idealization of the equilibrium state of a thin inextensible elastic wire which is straight at rest, proposed by D. Bernoulli to L. Euler in 1744. If $p=2$ and $\mu\neq 0$, minimizers of (2.2) are elasticae having the shape of a circle at rest (i.e., when no external forces act on the curve). Thus, for simplicity, in this paper stationary curves of (2.2) will be called, in general, *p-elasticae*, although the precise meaning is given bellow in our Definition .

More concretely, within the space of smooth regular curves in $\mathbb{E}^3_r$ (with non-null velocity and acceleration, if $r=1$) defined on a fixed interval $[a_o,a_1]$, we shall denote by $\Omega^r_{p_op_1}$ the subspace of those curves joining two fixed points p_o and p_1 of $\mathbb{E}^3_r$ verifying that $(\varepsilon_2\langle\gamma'',\gamma''\rangle)^{\frac{1}{2}}>\mu$, that is, for any choice of $r\in\{0,1\}$ we define

$$\begin{aligned}\Omega^r_{p_op_1}=\Big\{\;&\beta:[a_o,a_1]\to\mathbb{E}^3_r\,;\,\beta(a_i)=p_i,\,i\in\{0,1\},\,\langle\beta'(t),\beta'(t)\rangle\neq 0,\\ &\langle\beta''(t),\beta''(t)\rangle\neq 0,\,\big(\varepsilon_2\langle\beta''(t),\beta''(t)\rangle\big)^{\frac{1}{2}}>\mu,\,\forall t\in[a_o,a_1]\Big\}\,,\end{aligned}$$

where $p_i\in\mathbb{E}^3_r$, $i\in\{0,1\}$ are arbitrary given points of $\mathbb{E}^3_r$. Now, for a given $l\in\mathbb{R}$, $l>0$, $\Omega^{rl}_{p_op_1}$ will denote the subspace of curves with length l, i.e.

$$\Omega^{rl}_{p_op_1}=\Big\{\beta\in\Omega^r_{p_op_1}\,;\,\mathrm{length}(\beta)=l\Big\}\,.$$

Definition 2.1. For given $p_o,p_1\in\mathbb{E}^3_r$, $r\in\{0,1\}$, and $l,p,\mu\in\mathbb{R}$, $l>0,p\neq 0,1$, in this note, a *(pinned) p-elastic curve* or, simply, *p-elastica* will be understood as an stationary curve of $\mathscr{F}^\mu_{p|\Omega^{rl}_{p_op_1}}$, the restriction of $\mathscr{F}^\mu_p$ to the space of admissible curves of length l, $\Omega^{rl}_{p_op_1}$.

We also consider the subspace of unit speed parametrized curves

$$\Omega^{r*}_{p_op_1} = \left\{ \beta \in \Omega^r_{p_op_1}\,; \|\beta'\| = \varepsilon_1 \right\}.$$

We define now the functional

$$\begin{aligned} \theta :\ & \Omega^r_{p_op_1} \longrightarrow \mathbb{R}\,; \\ & \theta(\gamma) := \int_{a_o}^{a_1} \left(\varepsilon_2 \langle \gamma''(t), \gamma''(t)\rangle - \mu\right)^p \, dt\,, \end{aligned} \tag{2.3}$$

then on the space of unit speed curves $\Omega^{r*}_{p_op_1}$, we have $\theta(\gamma) = \mathscr{F}^\mu_p(\gamma)$. Since the formula for the curvature with respect to an arbitrary parameter t is not simple, the energy in $\Omega^r_{p_op_1}$ given by the last term of (2.2) has a complicated expression. So θ can be seen as a simplification of $\mathscr{F}^\mu_p$ on $\Omega^{r*}_{p_op_1}$ from which one has that a curve γ parametrized by the arc-length is p-elastica, that is, an stationary curve of $\mathscr{F}^\mu_{p|\Omega^{rl}_{p_op_1}}$ (with $l = a_1 - a_o$), if and only if, γ is an stationary curve of $\theta_{|\Omega^{r*}_{p_op_1}}$. We therefore can carry out our analysis on $\theta_{|\Omega^{r*}_{p_op_1}}$. So, the p-elasticae problem boils down to minimize or, more generally, to find the stationary curves of (2.3) subject to the constraint $\langle \gamma'(t), \gamma'(t)\rangle = \varepsilon_1$. This constraint reads $\psi(\gamma, \gamma', t) = 0$, where $\psi(\gamma, \gamma', t) = \langle \gamma'(t), \gamma'(t)\rangle - \varepsilon_1 = \|\gamma'(t)\|^2 - \varepsilon_1$. One version of the Lagrange multiplier principle for non-holonomic constraints [11] says that a stationary curve of θ on $\Omega^{r*}_{p_op_1}$ is a stationary curve of the functional $\widetilde{\mathscr{F}^\mu_p}$ given by

$$\begin{aligned} \widetilde{\mathscr{F}^\mu_p} :\ & \Omega^r_{p_op_1} \longrightarrow \mathbb{R}\,; \\ & \widetilde{\mathscr{F}^\mu_p}(\gamma) := \int_{a_o}^{a_1} \left(\left(\varepsilon_2 \langle \gamma''(t), \gamma''(t)\rangle\right)^{\frac{1}{2}} - \mu \right)^p + \Lambda(t) \left(\langle \gamma'(t), \gamma'(t)\rangle - \varepsilon_1 \right) dt\,, \end{aligned} \tag{2.4}$$

where $\varepsilon_1, \varepsilon_2$ are the causal characters of γ', γ'', respectively, and $\Lambda(t)$ is a pointwise multiplier constraining speed.

3. Generalized Ermakov-Milne-Pinney Equation

Now, we are going to relate the curvatures $\kappa(s)$ of an arc-length p-elasticae with solutions of certain GEMP equations with constant coefficients. More concretely, if we define the function $\phi(s)$ as

$$\phi(s) = (\kappa(s) - \mu)^{p-1}\,, \tag{3.1}$$

$\kappa(s)$ being the curvature of a unit speed stationary curve of $\mathscr{F}^\mu_{p|\Omega^{rl}_{p_op_1}}$, $\gamma(s)$, we will see that $\phi(s)$ satisfies a GEMP with constant coefficients of the type

$$\ddot{\phi}(s) + \alpha\phi(s) + \frac{\delta}{\phi^3(s)} = \eta\phi^a(s) + \omega\phi^{a-1}(s) - \frac{\varepsilon_1\varepsilon_2}{a}\phi^{2a-1}(s) + \sigma\,, \tag{3.2}$$

for certain values of the parameters $a, \alpha, \delta, \eta, \omega, \sigma \in \mathbb{R}$, and where overdot means derivative with respect to the arc-length parameter s.

Proposition 3.1. *For given $r \in \{0,1\}$ and $p_o, p_1 \in \mathbb{E}^3_r$, let $\gamma(s) \in \Omega^{rl}_{p_op_1}$ $(l = a_1 - a_0)$ be a unit speed immersed curve. Denote its curvature by $\kappa(s)$ and by ε_i, $i \in \{1,2,3\}$ the causal characters of its Frenet frame (see §2). Assume that $\gamma(s)$ is a p-elastica, i.e., a critical curve of $\mathscr{F}^\mu_{p|\Omega^{rl}_{p_op_1}}$ with*

$p\in\mathbb{R}-\{0,1\}$. *Then the function $\phi(s)$ defined in (3.1) is a solution of the following GEMP equation with constant coefficients*

$$\ddot{\phi}+\varepsilon_1\varepsilon_2\mu^2\phi-\varepsilon_2\varepsilon_3\frac{e^2}{\phi^3}=-\varepsilon_1\varepsilon_2\frac{p-1}{p}\phi^{\frac{p+1}{p-1}}-\varepsilon_1\varepsilon_2\mu\frac{2p-1}{p}\phi^{\frac{p}{p-1}}+\varepsilon_1\varepsilon_2\lambda\frac{1}{p}(\phi^{\frac{1}{p-1}}+\mu)\,, \quad (3.3)$$

for some $e,\lambda\in\mathbb{R}$.

Proof. As we have said at the end of last section, if $\gamma:[a_o,a_1]\to\mathbb{E}^3_r$ is a p-elastica, then it has to be an stationary curve of the functional $\widetilde{\mathscr{F}_p^\mu}$, (2.4). Since $\widetilde{\mathscr{F}_p^\mu}$ acts on $\Omega^r_{p_op_1}$ whose curves are not necessarily arc-length parametrized, at this stage we are going to denote the curve parameter by t, $\gamma(t)$. Then, if W is a vector field along γ, that is, an infinitesimal variation of the curve, we have

$$\partial\widetilde{\mathscr{F}_p^\mu}(W)=\frac{\partial}{\partial\varepsilon}\widetilde{\mathscr{F}_p^\mu}(\gamma+\varepsilon W)\big|_{\varepsilon=0}=0\,,$$

that is,

$$\begin{aligned}0&=\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\int_{a_o}^{a_1}\Big((\varepsilon_2\langle(\gamma+\varepsilon W)'',(\gamma+\varepsilon W)''\rangle)^{\frac{1}{2}}-\mu\Big)^p+\Lambda(t)\left(\langle(\gamma+\varepsilon W)',(\gamma+\varepsilon W)'\rangle-\varepsilon_1\right)dt\\&=\int_{a_o}^{a_1}\left\langle\varepsilon_2 p\frac{\Big((\varepsilon_2\langle\gamma'',\gamma''\rangle)^{\frac{1}{2}}-\mu\Big)^{p-1}}{(\varepsilon_2\langle\gamma'',\gamma''\rangle)^{\frac{1}{2}}}\gamma'',W''\right\rangle+2\langle\Lambda(t)\gamma',W'\rangle\,dt\,.\end{aligned} \quad (3.4)$$

Therefore, after integrating by parts twice we obtain

$$0=\int_{a_o}^{a_1}\langle\mathscr{E}(\gamma),W\rangle\,dt+\mathscr{B}(W,\gamma)\Big|_{a_o}^{a_1},$$

where we are denoting

$$\mathscr{E}(\gamma)=\varepsilon_2 p\left(\frac{\Big((\varepsilon_2\langle\gamma'',\gamma''\rangle)^{\frac{1}{2}}-\mu\Big)^{p-1}}{(\varepsilon_2\langle\gamma'',\gamma''\rangle)^{\frac{1}{2}}}\gamma''\right)''-2(\Lambda(t)\gamma')', \quad (3.5)$$

and, where the *boundary term* is given by

$$\begin{aligned}\mathscr{B}(W,\gamma)&=\left\langle\varepsilon_2 p\frac{\Big((\varepsilon_2\langle\gamma'',\gamma''\rangle)^{\frac{1}{2}}-\mu\Big)^{p-1}}{(\varepsilon_2\langle\gamma'',\gamma''\rangle)^{\frac{1}{2}}}\gamma'',W'\right\rangle\\&\quad-\left\langle\varepsilon_2 p\left(\frac{\Big((\varepsilon_2\langle\gamma'',\gamma''\rangle)^{\frac{1}{2}}-\mu\Big)^{p-1}}{(\varepsilon_2\langle\gamma'',\gamma''\rangle)^{\frac{1}{2}}}\gamma''\right)'-2\Lambda(t)\gamma',W\right\rangle.\end{aligned} \quad (3.6)$$

Notice that, since we are working with curves immersed in $\Omega^r_{p_op_1}$, then any variational field along γ must vanish at p_o and p_1. And so, the second part of (3.6) vanishes when evaluated at a_o and a_1. Moreover, under suitable boundary conditions, the boundary term vanishes. Anyway, if γ happens to be a critical curve in $\mathbb{E}^3_r$ of $\widetilde{\mathscr{F}_p^\mu}$, under any boundary conditions, then by standard arguments we have that $\mathscr{E}(\gamma)=0$ must be satisfied which are, of course, the Euler-Lagrange equations of the problem. Now, we know that $\gamma(s)$ is parametrized by the arc-length s. Hence, denoting by $\{T,N,B\}$ its Frenet frame, the condition $\mathscr{E}(\gamma)=0$ amounts to

$$\langle \mathscr{E},T\rangle = 0\,, \quad \langle \mathscr{E},N\rangle = 0\,, \quad \langle \mathscr{E},B\rangle = 0\,. \tag{3.7}$$

Thus, combining (3.5), (3.7) and the Frenet equations (2.1), one can see after long direct computations that the Euler-Lagrange equations can be expressed in terms of the curvature and torsion of γ in the following way

$$-\varepsilon_1 p\left(\kappa\frac{d}{ds}\left((\kappa-\mu)^{p-1}\right)+\frac{1}{2}\kappa_s\,(\kappa-\mu)^{p-1}\right) = \Lambda'\,, \tag{3.8}$$

$$\frac{d^2}{ds^2}\left((\kappa-\mu)^{p-1}\right)-(\kappa-\mu)^{p-1}\left(\varepsilon_1\varepsilon_2\kappa^2+\varepsilon_2\varepsilon_3\tau^2\right)-\frac{2}{p}\kappa\Lambda = 0\,, \tag{3.9}$$

$$\frac{d}{ds}\left(\tau\,(\kappa-\mu)^{2(p-1)}\right) = 0\,. \tag{3.10}$$

Now, integrating equation (3.8) we get

$$\Lambda(s) = \frac{1}{2}\varepsilon_1\left((\kappa-\mu)^p+\lambda\right)-\varepsilon_1\, p\,\kappa\,(\kappa-\mu)^{p-1}\,, \tag{3.11}$$

where λ is a constant of integration. Therefore, substituting the value of Λ given by (3.11) in (3.9) and (3.10), we see that the equations $\mathscr{E}(\gamma)=0$ boil down to

$$\frac{d^2}{ds^2}\left((\kappa-\mu)^{p-1}\right)+(\kappa-\mu)^{p-1}\left(\varepsilon_1\varepsilon_2\kappa^2-\varepsilon_2\varepsilon_3\tau^2\right)-\varepsilon_1\varepsilon_2\frac{1}{p}\kappa\left((\kappa-\mu)^p+\lambda\right) = 0\,, \tag{3.12}$$

$$\frac{d}{ds}\left((\kappa-\mu)^{2(p-1)}\,\tau\right) = 0\,, \tag{3.13}$$

which must be satisfied by a p-elastica. Now, introducing $\phi(s)$ as defined by (3.1) in (3.12) and (3.13), we obtain, after some direct manipulations, that the above equations (3.12) and (3.13) become

$$\phi^2\tau = e\,, \quad e\in\mathbb{R}\,, \tag{3.14}$$

$$\ddot{\phi}+\varepsilon_1\varepsilon_2\mu^2\phi-\varepsilon_2\varepsilon_3\frac{e^2}{\phi^3} = -\varepsilon_1\varepsilon_2\frac{p-1}{p}\phi^{\frac{p+1}{p-1}}-\varepsilon_1\varepsilon_2\mu\frac{2p-1}{p}\phi^{\frac{p}{p-1}}+\varepsilon_1\varepsilon_2\lambda\frac{1}{p}\left(\phi^{\frac{1}{p-1}}+\mu\right). \tag{3.15}$$

This finishes the proof. □

Notice that the expression (3.2) is obtained from (3.15) after the following choice of parameters

$$a=\frac{p}{p-1}\,, \quad \alpha=\varepsilon_1\varepsilon_2\mu^2\,, \quad \delta=-\varepsilon_2\varepsilon_3 e^2\,,$$
$$\omega=\varepsilon_1\varepsilon_2\frac{\lambda}{p}\,, \quad \eta=-\varepsilon_1\varepsilon_2\mu\frac{2p-1}{p}\,, \quad \sigma=\varepsilon_1\varepsilon_2\frac{\lambda}{p}\mu\,. \tag{3.16}$$

Conversely, choose $r\in\{0,1\}$ and take three numbers $\{\varepsilon_i, i=1,2,3\}$ satisfying: *i*) at most one of them is negative, *ii*) $\varepsilon_i=\pm1$, and *iii*) $\varepsilon_1\varepsilon_2\varepsilon_3=(-1)^r$. For every $p\neq 0,1$, consider real constants $\mu,\omega,\delta\in\mathbb{R}$ such that $-\varepsilon_2\varepsilon_3\delta\geq 0$. Take any solution $\phi(s)>0$ of the following GEMP equation with constant coefficients

$$\ddot{\phi}(s)+\varepsilon_1\varepsilon_2\mu^2\phi(s)+\frac{\delta}{\phi^3(s)}=-\varepsilon_1\varepsilon_2\,\mu\frac{a+1}{a}\phi^a(s)+\omega\phi^{a-1}(s)-\frac{\varepsilon_1\varepsilon_2}{a}\phi^{2a-1}(s)+\mu\omega\,, \quad (3.17)$$

where $a=\frac{p}{p-1}$. Now, formula (3.1) defines a function $\kappa(s)$, and then another function $\tau(s)$ is defined by (3.14), $e\in\mathbb{R}$ being a constant satisfying $e^2=-\varepsilon_2\varepsilon_3\delta$. Then, up to congruences in $\mathbb{E}^3_r$, there exists a unique unit speed Frenet curve $\gamma(s)$ having $\kappa(s)$ and $\tau(s)$ as curvature and torsion, respectively, and $\{\varepsilon_i, i=1,2,3\}$ as the causal characters of its Frenet frame. Hence, we obtain the following proposition

Proposition 3.2. *Under the previous conditions, the curve $\gamma(s)$ of $\mathbb{E}^3_r$ constructed out of a solution of (3.17) verifies the Euler-Lagrange equations* (3.12)-(3.13) *of the energy functional $\mathscr{F}^\mu_p$ defined in (2.2).*

Proof. Just combine (3.1), (3.14) and (3.17) to check that $\kappa(s)$ and $\tau(s)$ satisfy the Euler-Lagrange equations (3.12) and (3.13). □

Finally, we illustrate our result with a few examples.

Example 3.1. *Bending energy functional*. This corresponds to taking $p=2$ and $\mu=0$ in (2.2) and what we obtain is a functional measuring the *bending energy* of the curve, according to the Euler-Bernoulli classical model. Critical curves of the bending energy in semi-Riemannian manifolds are called *elastica*. In $\mathbb{E}^3$ elasticae's curvatures form a 3-parameter family obtained as the solutions of (3.12) and (3.13). They can be expressed in terms of the Jacobi elliptic functions

$$\kappa^2(s)=\kappa_o\left(1-\frac{h^2}{\nu^2}\,\mathrm{sn}^2\left(\frac{\kappa_o\, s}{2\nu},h\right)\right) \quad (3.18)$$

where sn denotes the Jacobi elliptic sine, κ_o is the maximum of the curvature, and $0\le h\le\nu\le 1$ (for details, see [17]). A similar result can be obtained in the Minkowski 3-space. Thus, from (3.1) and Proposition 3.1, we see that $\phi(s)=\kappa(s)$ given by (3.18) is a solution of the following generalized EMP equation

$$\ddot{\phi}(s)-\frac{1}{2}\lambda\phi(s)=\frac{e^2}{\phi^3(s)}-\frac{1}{2}\phi^3(s)\,.$$

Observe that above $\phi(s)=\kappa(s)$ has another interpretation in Fluid Dynamics, since it corresponds to the evolution velocity of an elastica under its Binormal flow, so spanning a *Hasimoto surface* in $\mathbb{E}^3$. Moreover, from (3.13) and the elastica's curvature (3.18) one obtains the elastica's torsion. Now, combining both as indicated by the *Hasimoto transformation*, one obtains a complex wave function providing soliton solutions of the *nonlinear Schrödinger equation* (for more details see, for instance, [12]).

Example 3.2. *Blaschke's energy type functionals*. In 1930, Blaschke studied the variational problem determined by taking $p=\frac{1}{2}$ and $\mu=0$ in (2.2), for variations of curves in the Euclidean 3-space $\mathbb{E}^3$ with no penalty on the length of the curves, and he found that the extremals are a kind of generalized helices, [6] p. 38. In [3] we have analyzed an extension of this problem by considering curves in spaces of constant sectional curvature which are critical for (2.2) when $p=\frac{1}{2}$ and $\mu\neq 0$.

In particular, for any $\mu \in \mathbb{R}$, $p = \frac{1}{2}$ and when $\lambda = 0$, we have that (3.12) and (3.13) can be integrated by using standard procedures and some known formulae in [?], so obtaining the curvature $\kappa(s)$ of the critical curves for $\mathscr{F}^{\mu}_{\frac{1}{2}}$ in $\mathbb{E}^3_r$. Then, $\phi(s) = \frac{1}{\sqrt{\kappa(s)-\mu}}$ is got from (3.1). Thus, if we define $\Delta = \alpha(\varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 e^2) - (2d + \varepsilon_1\mu)^2$ (with the notation introduced in (3.16)) we can see that the following functions

- If $\Delta \neq 0$ and $\alpha \neq 0$,

$$\phi^2(s) = \frac{2\varepsilon_2 d + \varepsilon_1\varepsilon_2\mu - \sqrt{|\Delta|} f(2\sqrt{|\alpha|}s)}{\alpha},$$

where, $f(x) = \sinh x$, if $\alpha < 0$ and $\Delta > 0$; $f(x) = \cosh x$, if $\alpha < 0$ and $\Delta < 0$; and $f(x) = \sin x$, if $\alpha > 0$ and $\Delta < 0$.

- If $\Delta = 0$ and $\alpha < 0$

$$\phi^2(s) = \frac{(4\varepsilon_2 d + 2\varepsilon_1\varepsilon_2\mu)\exp 2\sqrt{-\alpha}s - 1}{2\alpha};$$

- If $\Delta < 0$ and $\alpha = 0$,

$$\phi^2(s) = \frac{(4d + 2\varepsilon_1\mu)^2 s^2 + \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 e^2}{4\varepsilon_2 d + 2\varepsilon_1\varepsilon_2\mu};$$

- If $\Delta = \alpha = 0$,

$$\phi^2(s) = 2\sqrt{-\varepsilon_1\varepsilon_2 - \varepsilon_2\varepsilon_3 e^2}s,$$

comprise, according to Proposition 3.1, all non-constant possible solutions of the classical EMP equation with constant coefficients

$$\ddot{\phi}(s) + \varepsilon_1\varepsilon_2\mu^2\phi(s) - (\varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 e^2)\frac{1}{\phi^3(s)} = 0. \tag{3.19}$$

Observe that (3.19) is a particular instance of (1.1). It is well known that, by using a superposition principle, the general solution for the EMP equation (1.1) can be written as $x = (Ax_1^2 + 2Bx_1x_2 + Cx_2^2)^{\frac{1}{2}}$, where here A, B and C are real constants such that $AC - B^2 = \frac{h}{\mathscr{W}}$, $\mathscr{W}$ being the Wronskian of two independent solutions, x_1 and x_2, of the Schrödinger equation $x'' + q(t)x = 0$, [19]. Using this, the previous set of solutions $\phi(s)$ for (3.19) can also be obtained. Again, in this case the associated extremals evolve under their binormal flow with velocity $\phi(s)$ sweeping out surfaces with constant mean curvature in $\mathbb{E}^3_r$, [3].

4. First integrals

In this section we obtain first integrals of the GEMP equation (3.3). First, we notice that, if the curvature $\kappa(s)$ is not constant, we can apply Noether's theorem to derive first integrals of (3.12) and (3.13). In fact, assume that γ is an extremal of (2.2) and W is an infinitesimal symmetry. Then, the variation formula (3.4) continuous to hold for any intermediate value $a_o < t < a_1$, so that the boundary term (3.6) is constant along γ. In particular, for a constant vector field W (translational symmetry) we have

$$\left\langle \varepsilon_2 \left(\frac{\left((\varepsilon_2 \langle \gamma'', \gamma'' \rangle)^{\frac{1}{2}} - \mu \right)^{p-1}}{(\varepsilon_2 \langle \gamma'', \gamma'' \rangle)^{\frac{1}{2}}} \gamma'' \right)' - \frac{2}{p} \Lambda(t) \gamma', W \right\rangle = \text{constant}.$$

Letting W range over all translations, we get

$$\varepsilon_2 \left(\frac{\left((\varepsilon_2 \langle \gamma'', \gamma'' \rangle)^{\frac{1}{2}} - \mu \right)^{p-1}}{(\varepsilon_2 \langle \gamma'', \gamma'' \rangle)^{\frac{1}{2}}} \gamma'' \right)' - \frac{2}{p} \Lambda(t) \gamma' = D, \tag{4.1}$$

for some constant vector field D. Now, if we take γ to be arc-length parametrized, one can take advantage of the Frenet equations (2.1) so that equation (4.1) becomes

$$J = \left(-\varepsilon_1 \kappa (\kappa - \mu)^{p-1} - \frac{2}{p} \Lambda \right) T + \frac{d}{ds} \left((\kappa - \mu)^{p-1} \right) N + \varepsilon_3 \tau (\kappa - \mu)^{p-1} B, \tag{4.2}$$

along $\gamma(s)$ for a constant vector field J. On the other hand, (3.13) can be easily integrated once to $(\kappa - \mu)^{2(p-1)} \tau = e$, $e \in \mathbb{R}$. Moreover, since J is constant, so is its length, $p^2 |J|^2 = d$, $d \in \mathbb{R}$, which in combination with (3.11) and (4.2) gives that

$$\begin{aligned} \dot{\kappa}^2 &= \frac{1}{p^2 (p-1)^2} (\kappa - \mu)^2 \Big(\varepsilon_2 (d - \varepsilon_1 \lambda^2)(\kappa - \mu)^{2(1-p)} - \varepsilon_2 \varepsilon_3 p^2 \tau^2 \\ &\quad - \varepsilon_1 \varepsilon_2 \left((p-1)(\kappa - \mu) + p\mu \right) \left((p-1)(\kappa - \mu) + p\mu - 2\lambda (\kappa - \mu)^{1-p} \right) \Big), \qquad (4.3) \\ \tau &= e (\kappa - \mu)^{2(1-p)}. \qquad (4.4) \end{aligned}$$

are first integrals of the Euler-Lagrange equations (3.12) and (3.13).

Finally, by combining the two formulae in (4.3) and (4.4) and substituting the value of $\phi = (\kappa - \mu)^{p-1}$ in the resulting expression, we obtain, after some direct long computations

$$\dot{\phi}^2 = h - \frac{\varepsilon_2 \varepsilon_3 e^2}{\phi^2} - \frac{\varepsilon_1 \varepsilon_2 \phi^2}{p^2} \left((p-1) \phi^{\frac{1}{p-1}} + p\mu \right) \left((p-1) \phi^{\frac{1}{p-1}} + p\mu - \frac{2\lambda}{\phi} \right), \tag{4.5}$$

for some constants $h, e, \lambda \in \mathbb{R}$. Hence, equation (4.5) is a first integral of (3.3).

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