Boundary Value Problems for the Helfrich Energy

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- W. Blaschke and G. Thomsen ($\sim$1920): The functional $\mathcal{W}$ is conformally invariant.

Modeling Biological Membranes

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- P. B. Canham (1970): Proposed the minimization of the Willmore energy as a possible explanation for the biconcave shape of red blood cells.

Modeling Biological Membranes

- W. Helfrich (1973): Based on liquid cristallography, suggested the extension

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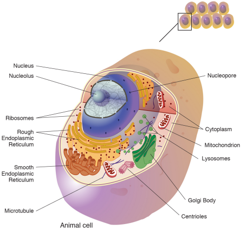

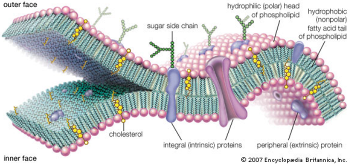

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Gauss-Bonnet Theorem

The total Gaussian curvature term only affects the boundary.

Euler-Lagrange Equation

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Theorem (Palmer & A. P., submitted)

The immersion $X : \Sigma \to \mathbb{R}^3$ is critical for the Helfrich energy $\mathcal{H}$ with respect to compactly supported variations if and only if Y^{c_o} is critical for

$$\mathcal{F}[Z] := \int_\Sigma \left(\|dZ\|^2 + 4c_o U(Z) \right) d\Sigma\,,$$

where $U(Z) := Z_4 - Z_5$.

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(Satisfied by axially symmetric discs.)

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- The surface must be a topological disc. Annular domains in circular biconcave discoids are critical for $\mathcal{H}$.

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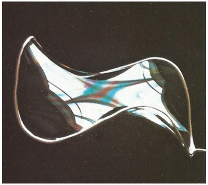

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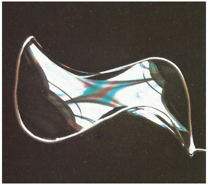

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The Euler-Helfrich Problem

The Euler-Helfrich energy is given by:

$$E[\Sigma] := \int_\Sigma \left(a\,[H + c_o]^2 + bK \right) d\Sigma + \oint_{\partial\Sigma} (\alpha\kappa^2 + \beta)\, ds\,,$$

where $\alpha > 0$ and $\beta > 0$.

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where J is a vector field along $\partial\Sigma$ defined by

$$J := 2\alpha T'' + \left(3\alpha\kappa^2 - \beta\right) T\,.$$

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Boundary Curves (Palmer & A. P., 2021)

Let $X : \Sigma \to \mathbb{R}^3$ be an equilibrium with $H + c_o \equiv 0$. Then, each boundary component C is a simple and closed critical curve for

$$F[C] \equiv F_{\mu,\lambda}[C] := \int_C \left([\kappa + \mu]^2 + \lambda \right) ds\,,$$

where $\mu := \pm b/(2\alpha)$ and $\lambda := \beta/\alpha - \mu^2$.

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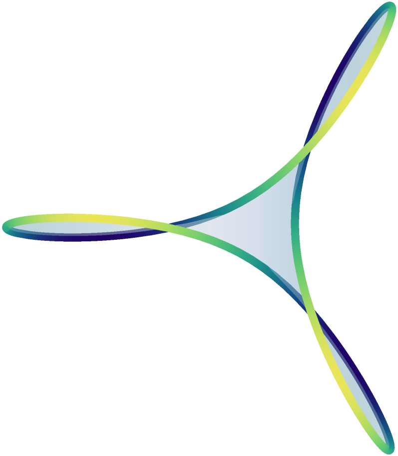

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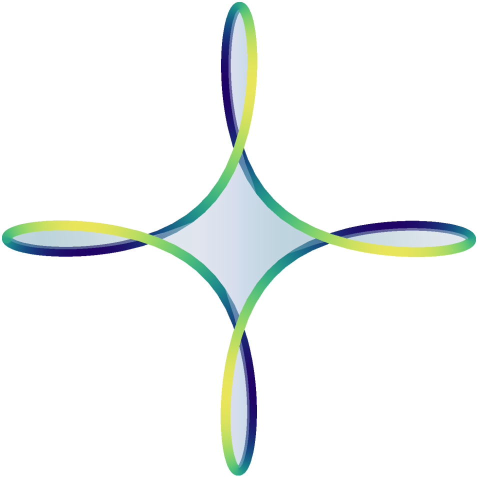

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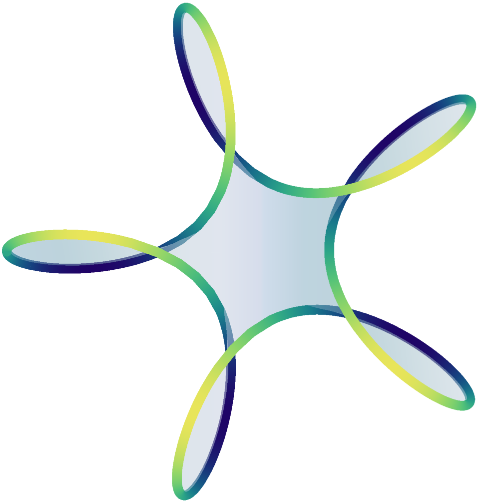

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THE END

- B. Palmer and A. Pámpano, Minimizing Configurations for Elastic Surface Energies with Elastic Boundaries, *Journal of Nonlinear Science*, **31-23** (2021).

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- B. Palmer and A. Pámpano, The Euler-Helfrich Functional, *submitted*.

Thank You!