A New Variational Characterization of Invariant CMC Surfaces

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Geometry Seminar
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- Wente (1984): Found an immersed torus with CMC.

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- Note that γ is the curve everywhere orthogonal to ξ. (It is not necessarily planar, i.e., it may not be contained in a totally geodesic surface of $M_r^3(\rho)$.)

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- Then, the Frenet equations,

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- From the Fundamental Theorem for Curves, this is all what we need.

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- We call this energy the extended Blaschke's energy, since in 1930 Blaschke studied the case $\mu = 0$ in $\mathbb{R}^3$.

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3. Finally, we combine this with H constant to obtain an ODE in $P(\kappa)$ which can be explicitly solved.

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- Biconservative Surfaces
  (Montaldo & A. P., to appear)

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acting on the space of non-null curves, with non-null acceleration, immersed in $M_r^3(\rho)$.

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- In $\mathbb{R}^2$, $e = 0$ and critical curves are roulettes of conic foci.

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4. We construct the binormal evolution surface **(Garay & A. P., 2016)**

$$S_\gamma := \{x(s,t) := \psi_t\,(\gamma(s))\} \,.$$

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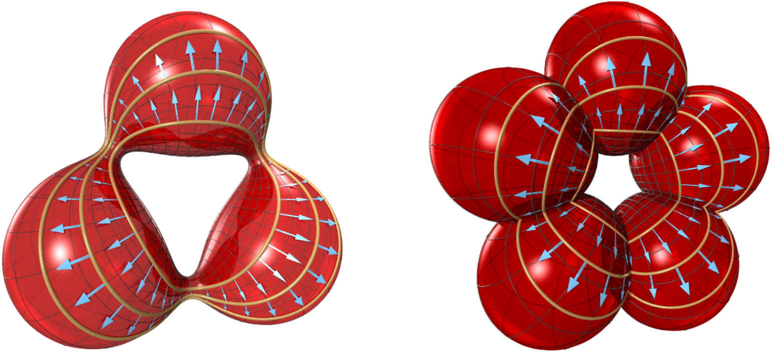
Binormal Evolution Surfaces in $\mathbb{S}^3(\rho)$



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- A planar curve has $\tau = 0$ and it lies in $\mathbb{S}^2(\rho)$.
- The curvature of a planar critical curve for $\mathbf{\Theta}_\mu$ in $\mathbb{S}^2(\rho)$ is:

$$\kappa_d(s) = \frac{\rho + \mu^2}{2d + \mu - \sqrt{4d^2 + 4\mu d - \rho}\,\sin(2\sqrt{\rho + \mu^2}\, s)} + \mu\,,$$

for $d \geq (-\mu + \sqrt{\mu^2 + \rho})/2$.

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3. A Hopf torus ($\kappa(s) = -|H| + \sqrt{H^2 + \rho}$)

$$\mathbb{S}^1\left(\sqrt{\rho+\kappa^2}\right) \times \mathbb{S}^1\left(\frac{\sqrt{\rho}}{\kappa}\sqrt{\rho+\kappa^2}\right) .$$

Local Classification

Theorem (Arroyo, Garay & A. P., 2019)

Locally, a rotational surface of CMC H in $\mathbb{S}^3(\rho)$ is congruent to a piece of:

1. The equator ($\kappa(s) = H = 0$).
2. A totally umbilical sphere ($\kappa(s) = |H| \neq 0$).
3. A Hopf torus ($\kappa(s) = -|H| + \sqrt{H^2 + \rho}$)

$$\mathbb{S}^1\left(\sqrt{\rho + \kappa^2}\right) \times \mathbb{S}^1\left(\frac{\sqrt{\rho}}{\kappa}\sqrt{\rho + \kappa^2}\right) .$$

4. A binormal evolution surface ($\kappa(s) = \kappa_d(s)$ and $|\mu| = |H|$).

Closed Critical Curves in $\mathbb{S}^2(\rho)$

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Theorem (Arroyo, Garay & A. P., 2019)

There exist non-trivial closed critical curves in $\mathbb{S}^2(\rho)$, for any value of μ.

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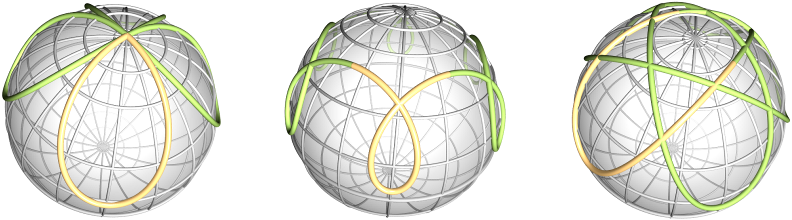

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Closed and Simple Critical Curves in $\mathbb{S}^2(\rho)$

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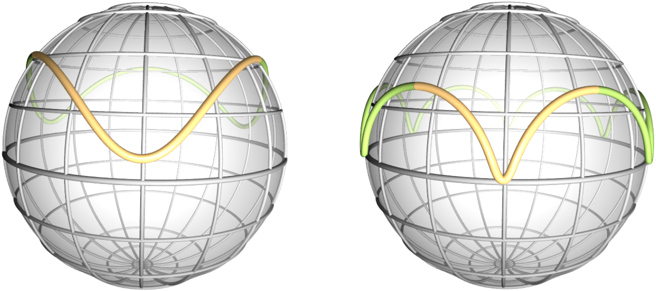

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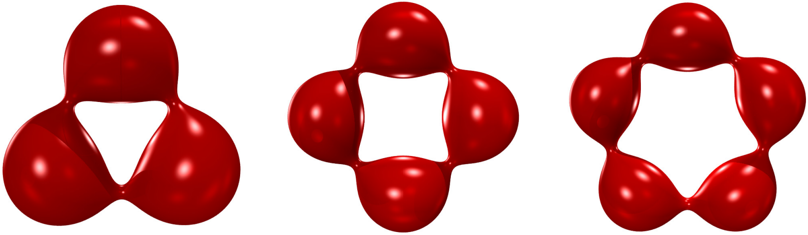
CMC Tori in $\mathbb{S}^3(\rho)$



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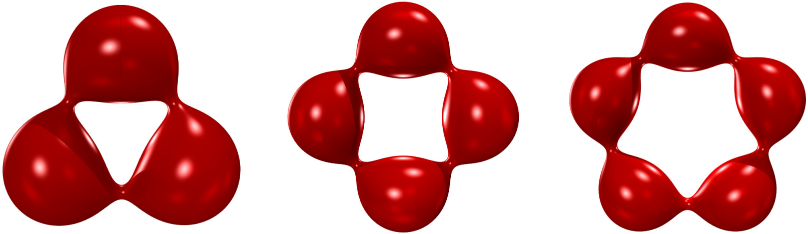

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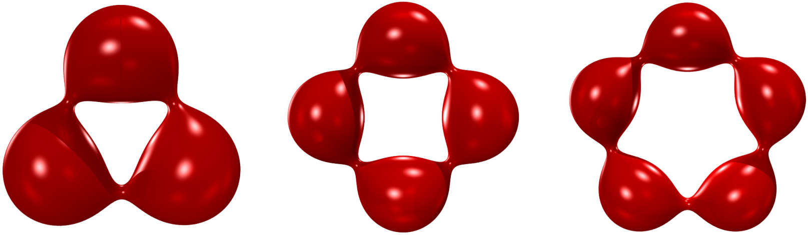

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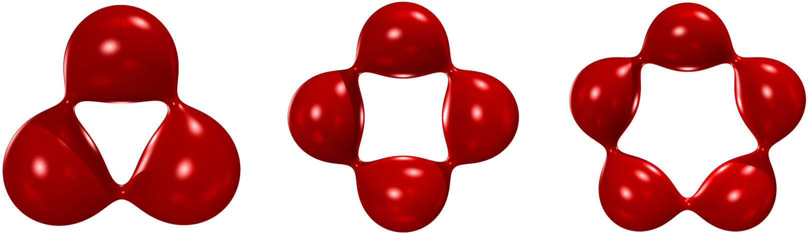

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- Coincides with previous results of Perdomo and Ripoll.
- Verify the Lawson's conjecture (proved by Brendle in 2013).
- After Pinkall-Sterling's conjecture (proved by Andrews-Li in 2015), these are all embedded CMC tori.

THE END

- J. Arroyo, O. J. Garay and A. Pámpano, Constant Mean Curvature Invariant Surfaces and Extremals of Curvature Energies, *J. Math. Anal. Appl.* **462-2** (2018), 1644-1668.
- J. Arroyo, O. J. Garay and A. Pámpano, Delaunay Surfaces in $\mathbb{S}^3(\rho)$, *Filomat* **33-4** (2019), 1191-1200.

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Thank You!