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WILLMORE-LIKE ENERGIES AND ELASTIC CURVES WITH POTENTIAL

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Abstract. We study invariant Willmore-like tori in total spaces of Killing submersions. In particular, using a relation with elastic curves with potentials in the base surfaces, we analyze Willmore tori in total spaces of Killing submersions. Finally, we apply our findings to construct foliations of these total spaces by constant mean curvature Willmore tori.

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1. Introduction

These notes are a printed version of the talk given by the author at the International Conference on Geometry, Integrability and Quantization held in Varna in June 2019. The purpose of the talk was to present some results included in the work [3]. Here, ideas and arguments are only sketched while proofs are omitted. Interested readers are going to be referred to [3], for a complete and more general treatment.

One of the most important tools to understand the extrinsic geometry of an immersed surface into an ambient space is, probably, the *mean curvature*. This term was first coined by S. Germain who considered the integral of the squared mean curvature as a proposal to measure the free energy controlling the physical system associated with an elastic plate, [7].

This energy is usually called *total squared curvature* or *bending energy*. In particular, for a surface S immersed into the Euclidean 3-space $\mathbb{R}^3$, the bending energy is given by

$$\mathcal{W}(S) = \int_S H^2 \, dA$$

where the mean curvature of S is denoted by H. The variational problem associated to this bending energy was studied in the decade of 1920 by Blaschke's school obtaining, among many other things, the associated Euler-Lagrange equation by means of computing the first variation formula, [13]. Moreover, they also discovered the nice property that the bending energy is conformal invariant, [4].

Nowadays, the functional $\mathcal{W}(S)$ is widely known as the *Willmore energy*, since the associated variational problem was reintroduced by T. J. Willmore in the decade of 1960, [14]. He investigated the minima for the bending energy within a given topological class and, as a particular case, in 1965, he proposed the following conjecture: *the Willmore energy of any smooth immersed torus is greater or equal* $2\pi^2$*, with equality for the Clifford torus*.

This conjecture has attracted the interest of many researchers and, therefore, in the last years there has been an intensive investigation about this subject. We highlight here the work [10], where F. C. Marques and A. Neves proved the Willmore conjecture.

The Willmore energy has also been generalized to other ambient spaces so that the conformal invariance property is preserved. For instance, given a surface S of an arbitrary Riemannian ambient 3-space M, the following energy

$$\mathcal{CW}(S) = \int_S \left(H^2 - \tau_e\right)\, dA$$

where τ_e is the *extrinsic scalar curvature* of S represents a generalization of the Willmore energy. This generalization is due to B.-Y. Chen, [5], and, as a consequence, the energy $\mathcal{CW}(S)$ is usually referred to as the *Chen-Willmore energy*.

If the ambient space M happens to have constant sectional curvature, above generalization is simply done by introducing a constant in the integrand. In this setting, it turns out that there exists a beautiful link between critical points of $\mathcal{CW}(S)$, i.e. *Chen-Willmore surfaces* (or, simply, *Willmore surfaces*) and elastic curves, which can be established using a symmetry reduction procedure, [12] (see also [11]).

Elastic curves have been classically studied by Galileo, the Bernoulli family, Euler, Kirchhoff, Born and many others. In fact, the study of elasticae brings together two classical subjects: the theory of curves and the mechanics of solids.

In 1691, J. Bernoulli formulated the problem of determining the bending deformation of rods. For ideal elastic rods, if there is no twist and the rod is bent in a plane then, as suggested by D. Bernoulli (nephew of J. Bernoulli), these rods should bend along a curve which minimizes the bending energy. Observe that for curves, γ, isometrically immersed in a Riemannian manifold, the mean curvature H is nothing but the geodesic curvature of γ, κ, and hence the bending energy $\mathcal{W}(S)$ reduces to

$$\mathcal{E}(\gamma) = \int_\gamma \kappa^2\, ds\,.$$

Critical curves of $\mathcal{E}(\gamma)$ are called *Euler-Bernoulli elasticae* or *elastic curves*.

Using this formulation, in his book of 1744, L. Euler described the possible qualitative types of elastic curves in $\mathbb{R}^2$, [6], although some partial results were already known to J. Bernoulli. This book of L. Euler represents one of the fundamental basis for the development of the Calculus of Variations. Thus, clearly, the study of elastic curves played an important role in this development. For a more detailed historical background, we refer the reader to [8].

In the last decades, elastic curves in different ambient spaces have been considered. Moreover, this notion has also been generalized introducing different modifications in the integrand of the energy $\mathcal{E}(\gamma)$.

In this paper, by considering elastic curves with potential on a surface, we are going to study invariant Willmore-like tori in total spaces of Killing submersions. In Section 2, we compute the first variation formula and the associated Euler-Lagrange equation for a closed Willmore-like surface in an arbitrary 3-space.

Then, in Section 3, we recall the basic facts about Killing submersions and the construction of invariant tori in their total spaces. Focusing on these spaces, in Section 4 we obtain a nice relation between invariant Willmore-like tori and elastic curves with potential.

Finally, this relation is used in Section 5 to study invariant Willmore tori in total spaces of Killing submersions. In particular, two different ways of foliating these total spaces by Willmore tori with constant mean curvature are analyzed.

2. Willmore-Like Energies in 3-Spaces

Let $(M, \langle\cdot,\cdot\rangle)$ be a 3-dimensional Riemannian manifold with Levi-Civita connection $\bar{\nabla}$, and S a closed surface, i.e. a compact surface without boundary. We will denote by Δ the *Laplacian* associated with the connection $\bar{\nabla}$, while Ric stands for the *Ricci curvature* of M.

For any isometric immersion of S in M, $\varphi : S \to M$, we denote by dA_φ the induced *area element* obtained via φ. Moreover, if we fix η to be a unit normal vector field of φ, we have that the *mean curvature vector* of φ can be written as

$$\vec{H}_\varphi = H_\varphi \eta \,,$$

where H_φ denotes the *mean curvature function* of φ. Let us denote by K_S the (intrinsic) *Gaussian curvature* of S endowed with the induced metric associated to φ and by R the *extrinsic Gaussian curvature*, i.e. the sectional curvature of M on the tangent plane of φ.

Then, in the space of isometric immersions of S in M (denoted throughout the paper by $I\,(S,M)$), we define for any smooth function $\Phi\in\mathcal{C}^\infty(M)$ the *Willmore-like energy* by

$$\mathcal{W}_\Phi\,(S)\equiv\mathcal{W}_\Phi\,(S,\varphi):=\int_S\left(H_\varphi^2+\Phi|_{\varphi(S)}\right)\,dA_\varphi\,. \tag{1}$$

A *variation* of $\varphi\in I(S,M)$ is a smooth map $\widetilde{\varphi}:S\times(-\varepsilon,\varepsilon)\to M$ where $\widetilde{\varphi}(p,0)\equiv\varphi_o(p)=\varphi(p)$, for all $p\in S$ and which satisfies that for any $\varsigma\in(-\varepsilon,\varepsilon)$, the map $\varphi_\varsigma\equiv\widetilde{\varphi}(-,\varsigma)$ belongs to $I(S,M)$. In this setting, there exists a vector field along φ,

$$V(p)\equiv V(p,0):=\widetilde{\varphi}_*\left(\frac{\mathrm{d}}{\mathrm{d}\varsigma}(p,\varsigma)\right)\Big|_{\varsigma=0}$$

which is called the *variation vector field* associated to $\widetilde{\varphi}$. Thus, we can identify the tangent space $T_\varphi\,(I(S,M))$ with the space of vector fields along φ and, consequently, we have

$$\delta\mathcal{W}_\Phi\,(S,\varphi)\,[V]=\frac{\mathrm{d}}{\mathrm{d}\varsigma}\Big|_{\varsigma=0}\left(\int_S\left(H_{\varphi_\varsigma}^2+\Phi\right)\,dA_{\varphi_\varsigma}\right)\,.$$

Then, we have the following characterization of Willmore-like surfaces.

Theorem 1 ([3]). *Let M be a 3-dimensional Riemannian manifold, $\Phi\in\mathcal{C}^\infty(M)$ a smooth function and S a closed surface isometrically immersed in M. Then, S is a Willmore-like surface, if and only if, the Euler-Lagrange equation*

$$\Delta H+H\left(2H^2-2K_S+2R+Ric(\eta,\eta)-2\Phi\right)+\eta\,(\Phi)=0 \tag{2}$$

is satisfied.

3. Tori in Total Spaces of Killing Submersions

Let M be a 3-dimensional Riemannian manifold as in previous section and consider a fixed surface B. A Riemannian submersion $\pi:M\to B$ of M over B is called a *Killing submersion* if its fibers are the trajectories of a complete unit Killing vector field, ξ. Fibers of Killing submersions are geodesics in M and form a foliation called the *vertical foliation*. Indeed, the Killing vector field ξ is sometimes referred to as the *vertical Killing* vector field.

Moreover, since ξ is a vertical unit Killing vector field, then it is clear that for any vector field, Z, on M, there exists a function τ_Z such that

$$\bar{\nabla}_Z\xi=\tau_Z Z\wedge\xi\,,$$

where $\wedge$ denotes the usual vector product. Indeed, as proved in [9], one can see that τ_Z does not depend on the vector field Z, so we have a function $\tau\in\mathcal{C}^\infty(M)$, the *bundle curvature*. Obviously, the bundle curvature is constant along fibers and, consequently, it can be seen as a function on the base surface $\tau \in \mathcal{C}^\infty(B)$ (denoted again by τ).

From now on, the 3-manifold M is going to be called the *total space* of the Killing submersion and the surface B, the *base surface*. It turns out that most of the geometry of a Killing submersion is encoded in a pair of functions defined on the base surface, namely, its *Gaussian curvature*, K_B, and the *bundle curvature*, τ. Therefore, it is usual to denote the total space of a Killing submersion by $M \equiv M(K_B, \tau)$.

A natural question that arises at this point concerns the existence of Killing submersions over a given surface B (with Gaussian curvature K_B) for a prescribed bundle curvature $\tau \in \mathcal{C}^\infty(B)$. For arbitrary Riemannian surfaces existence has been proven in [3]. In particular, if the base surface happens to be simply connected, then this result was first proven in [9]. In the same paper, uniqueness (up to isomorphisms) is also guaranteed under the assumption that the total space is also simply connected. Moreover, fibers can always be supposed to have finite length.

Let γ be an immersed curve in B, then the pre-image of γ via the Killing submersion π, $S_\gamma := \pi^{-1}(\gamma)$ is a surface isometrically immersed in M. Indeed, it is clear that S_γ is invariant under the one-parameter group of isometries associated with the vertical Killing vector field, ξ, $\mathcal{G} = \{\psi_t\,;\, t \in \mathbb{R}\}$. Furthermore, any $\mathcal{G}$-invariant surface in M, S, is obtained by this construction for some curve γ of B. These surfaces S_γ are called *vertical tubes* (or, also, *vertical cylinders*) shaped on the curve γ.

Assume that γ is parametrized by its arc-length, s. Then, any horizontal lift of γ, $\bar{\gamma}$, is also arc-length parametrized. Now, using as coordinate curves the horizontal lifts of γ and the fibers of the Killing submersion, vertical tubes S_γ can be parametrized by

$$x(s,t) = \psi_t\left(\gamma(s)\right) .$$

Notice that S_γ is embedded if γ is a simple curve, and it is a torus when γ is closed and $\mathcal{G} \cong \mathbb{S}^1$ is a circle group. As mentioned before, the second condition ($\mathcal{G} \cong \mathbb{S}^1$) can always be assumed after suitable quotients, if necessary. However, even in this case, horizontal lifts of γ, $\bar{\gamma}$, may not be closed due to the non-trivial holonomy (for an example, see [1]).

Finally, as a consequence of the parametrization of these vertical tubes, we have that S_γ are always flat ($K_{S_\gamma} \equiv 0$). Moreover, the mean curvature function of these surfaces, H, is closely related to the curvature function of the cross sections as the following formula shows (for details, see [2])

$$H = \frac{1}{2}\left(\kappa \circ \pi\right) , \tag{3}$$

where κ denotes the geodesic curvature of γ in the base surface B.

4. Link with Elasticae with Potential

Consider a potential $\phi \in \mathcal{C}^\infty(B)$ defined on a surface B. We define the *bending energy with potential* by

$$\mathcal{E}_\phi(\gamma) = \int_\gamma \left(\kappa^2 + 4\phi\right) ds \tag{4}$$

acting on the space of closed curves of the surface B. Note that the choice of 4ϕ in the integrand comes from consistency purposes that are going to be clear later.

If the potential ϕ is constant, then the energy $\mathcal{E}_\phi$ is just the classical bending energy with a restriction on the length of the curves, due to a Lagrange multipliers principle. Therefore, extremal curves of $\mathcal{E}_\phi$, (4), for a general $\phi \in \mathcal{C}^\infty(B)$ are going to be called *elastic curves with potential*.

Moreover, if we compute the first variation formula by standard arguments involving integration by parts, we get that these extremal curves are characterized by the following Euler-Lagrange equation

$$2\kappa_{ss} + \kappa^3 + 2\left(K_B - 2\phi\right)\kappa + 4n(\phi) = 0 \tag{5}$$

where the subscript s means derivative with respect to the arc-length parameter, n is the unit (Frenet) normal of the curve γ in B and K_B is the Gaussian curvature of the surface B, as before.

Assume now that we have a Killing submersion over B, $\pi : M \to B$, with compact fibers, i.e. $\mathcal{G} \cong \mathbb{S}^1$. Then, using a symmetry reduction procedure, in [3] the following correspondence between elasticae with potential and Willmore-like tori was proven.

Theorem 2 ([3]). *Let $\pi : M \to B$ be a Killing submersion with compact fibers and consider an invariant potential Φ, i.e. $\Phi = \phi \circ \pi$. If γ is a closed curve in B, then its vertical torus $S_\gamma = \pi^{-1}(\gamma)$ is a Willmore-like surface for the potential Φ, if and only if, γ is an elastic curve with potential for ϕ, i.e. an extremal curve of*

$$\mathcal{E}_\phi(\gamma) = \int_\gamma \left(\kappa^2 + 4\phi\right) ds\,.$$

We point out here that above theorem can be constructively proven in the following way. Let γ be a closed curve in B. Choose any function $\tau \in \mathcal{C}^\infty(B)$ and construct the Killing submersion with compact fibers $\pi : M \equiv M(K_B, \tau) \to B$ whose existence is guaranteed as explained above. Then, on the vertical torus shaped on γ, S_γ, the following formula holds

$$2R + Ric\,(\eta, \eta) = K_B\,.$$

Hence, combining this, (2) and (3) with $K_{S_\gamma} = 0$ (since S_γ is flat) we get the Euler-Lagrange equation (5), proving the statement.

5. Applications to Willmore Tori

Although, in general, the Willmore-like energy $\mathcal{W}_\Phi$, (1), does not coincide with the Chen-Willmore energy $\mathcal{CW}$, the situation for vertical tori shaped on closed curves of the base surface is different. Indeed, we have the following result.

Theorem 3 ([3]). *Let $\pi : M \equiv M(K_B, \tau) \to B$ be a Killing submersion with compact fibers. Then, a vertical torus S_γ is a Willmore surface in M, if and only if, S_γ is an extremal of the Willmore-like energy*

$$\mathcal{W}_{\tau^2}(S) = \int_S \left(H^2 + \tau^2\right) dA$$

where τ denotes the bundle curvature of π.

Moreover, combining Theorem 2 with Theorem 3, we have the following characterization of invariant Willmore tori in total spaces of Killing submersions.

Corollary 4 ([3]). *A vertical torus shaped on a closed curve γ, $S_\gamma = \pi^{-1}(\gamma)$, is a Willmore surface in the total space $M(K_B, \tau)$, if and only if, γ is an elastica with potential $4\tau^2$ in B, i.e. an extremal curve of*

$$\mathcal{E}_{\tau^2}(\gamma) = \int_\gamma \left(\kappa^2 + 4\tau^2\right) ds\,.$$

In what follows we are going to apply these results to the construction of foliations of total spaces of certain Killing submersions, consisting of Willmore tori with constant mean curvature.

5.1. Willmore Foliations of Orthonormal Frame Bundles

In this subsection, we consider frame bundles of surfaces as a particular case of Killing submersions. For a given surface B, we denote by FB its orthonormal frame bundle. The natural projection $\pi : FB \to B$ gives a principle bundle, whose geometry is, certainly, more closely related to B than that of any other principle bundle.

In this setting, after some computations (see formulas (33)-(37) of [3]), we get that the bundle curvature of $\pi : FB \to B$ seen as a Killing submersion is related with the Gaussian curvature of B, K_B, by

$$\tau^2 = \frac{1}{4}\left(K_B^2 \circ \pi\right).$$

Hence, we have the following result as a direct consequence of Theorem 3.

Corollary 5 ([3]). *Assume that γ is a closed curve in a surface B, with Gaussian curvature K_B. Then, $S_\gamma = \pi^{-1}(\gamma)$ is a Willmore torus in the orthonormal frame bundle FB, if and only if, γ is an elastic curve with potential for*

$$\mathcal{E}_{\frac{1}{4}K_B^2}(\gamma) = \int_\gamma \left(\kappa^2 + K_B^2\right) ds$$

acting on the space of closed curves in B.

We are going to use this corollary to construct surfaces whose orthonormal frame bundle is foliated by Willmore tori with constant mean curvature. First, observe that if we consider any compact rotational surface in $\mathbb{R}^3$, its associated orthonormal frame bundle admits a foliation by minimal Willmore tori. This minimal tori come from vertical lifts of the meridians of the surface, [3].

Now, in order to get foliations by non-minimal Willmore tori we propose the following construction. Let $f(s)$ be a smooth function defined on a real interval I and consider the *warped product surface* $S_f := I \times_f \mathbb{S}^1$. The group $\mathbb{S}^1$ acts trivially on S_f by isometries and the orbits of this action are called the *fibers* of S_f.

Let us consider the warped product surfaces S_f such that all their fibers are extremal curves for

$$\mathcal{E}_{\frac{1}{4}K_{S_f}^2}(\gamma) = \int_\gamma \left(\kappa^2 + K_{S_f}^2\right) ds\,,$$

K_{S_f} denoting the Gaussian curvature of S_f. This property involving the criticality of fibers completely describes the function $f(s)$ as shown in [3]. Moreover, it turns out, applying Corollary 5, that these warped product surfaces, S_f, give rise to orthonormal frame bundles admitting foliations by Willmore tori with constant mean curvature.

5.2. Willmore Foliations of General Killing Submersions

Finally, we are going to describe a different approach to foliate total spaces of more general Killing submersions. Consider a (classical) elastic curve $\gamma(s)$ in a surface B, i.e. $\gamma(s)$ is an elastic curve with constant potential, namely $\phi = \lambda/4$. In other words, it is an extremal curve of

$$\mathcal{E}_{\lambda/4}(\gamma) = \int_\gamma \left(\kappa^2 + \lambda\right) ds$$

for any $\lambda \in \mathbb{R}$.

Now, we would like to obtain all the potentials $\phi \in \mathcal{C}^\infty(B)$ for which $\gamma(s)$ is also an elastic curve with potential. In a neighborhood of $\gamma(s)$ in B, these potentials ϕ must have the following form

$$\phi(s,t) = \mathrm{e}^{t\kappa(s)+\varpi(s)} + \lambda$$

for any arbitrary function $\varpi(s)$ along $\gamma(s)$ (see [3] for details).
Then, we have

Theorem 6 ([3]). *Let $\pi : M \to B$ be a Killing submersion with compact fibers and bundle curvature, τ, satisfying*

$$4\tau^2 = \phi(s,t)$$

on a certain neighborhood of $\gamma(s)$. Then, $S_\gamma = \pi^{-1}(\gamma)$ is a Willmore torus in M.

We finish this subsection with an illustration. Take $B = \mathbb{R}^2 \setminus \{(0,0)\}$, the once punctured Euclidean plane (recall that $K_B \equiv 0$ in this case) and consider the family of circles centered in $(0,0)$ and with radius $t > 0$, namely, $\{C_t\,;\, t > 0\}$.
Then, as shown in [3], for any function $g \in \mathcal{C}^\infty(\mathbb{S}^1)$, the potentials

$$\phi(s,t) = g(s)t + \frac{1}{3t^2}$$

make the whole family of circles elasticae with potential. Furthermore, the following result is a clear consequence.

Proposition 7 ([3]). *There exists a Killing submersion $\pi : M(0,\tau) \to B$ with bundle curvature given by $4\tau^2 = \phi(s,t)$ which admits a foliation by Willmore tori with constant mean curvature.*

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