Blaschke's Variational Problem

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- **1744**: L. Euler described the shape of planar elasticae (partially solved by Jacob Bernoulli, 1692-1694).

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- **Case** $p > 2$: (Applications: Willmore-Chen submanifolds, string theories,...)

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- **Cases** $p = (n-2)/(n+1)$: Arise in the theory of biconservative hypersurfaces. (Montaldo & P., 2020, Montaldo, Oniciuc & P., 2022)

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The case $p = 1/2$ plays the role of the classical bending energy (when $p \in (0, 1)$) and its study can be faced resorting to elliptic functions and integrals (as the case $p = 2$).

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- If $\kappa = \kappa_o$ is constant, then critical curves are circles with $\kappa_o = \sqrt{\rho}$ (necessarily $\rho \geq 0$).
- If κ is nonconstant, we can obtain a first integral (a conservation law).

Curvatures of Critical Curves

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$$\kappa_d(s) = \frac{\rho}{2d - \sqrt{4d^2 - \rho}\,\sin\left(2\sqrt{\rho}\, s\right)},$$

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3. In the hyperbolic plane $\mathbb{H}^2(\rho)$: (the same with cosh).

Parameterization of (Spherical) Critical Curves

Let γ_d be a spherical critical curve for $\mathbf{\Theta}$ immersed in $\mathbb{S}^2(\rho)$ having curvature $\kappa = \kappa_d$, then,

$$\gamma_d(s) = \frac{1}{2\sqrt{\rho d \kappa}} \left(\sqrt{\rho}, \sqrt{4d\kappa - \rho} \sin \Psi, \sqrt{4d\kappa - \rho} \cos \Psi \right),$$

where (angular progression)

$$\Psi(s) = 2\sqrt{\rho d} \int \frac{\kappa^{3/2}}{4d\kappa - \rho} \, ds \, .$$

Recall that $d > \sqrt{\rho}/2$.

Geometric Properties

1. The trajectory of γ_d is contained in a domain bounded by two parallels in the half-sphere $x > 0$. It never meets the equator $x = 0$ nor the pole $(1, 0, 0)$.

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3. The trajectory of γ_d winds around the pole $(1, 0, 0)$ without going backwards.
4. The curve γ_d is closed if and only if

$$\Lambda(d) = 2\sqrt{\rho\, d} \int_0^{\varrho} \frac{\kappa^{3/2}}{4d\kappa - \rho}\, ds = 2q\pi \,,$$

for a rational number $q \in \mathbb{Q}$.

Closure Condition

Using the first integral to make a change of variable, we have

$$\Lambda(d) = 2\sqrt{\rho\, d} \int_{\beta}^{\alpha} \frac{\kappa}{(4d\kappa - \rho)\sqrt{\kappa(\alpha - \kappa)(\kappa - \beta)}}\, d\kappa = 2q\pi$$

where $\alpha > \beta$ are the (only) positive roots of $Q_d(\kappa) = \kappa^2 - 4d\kappa + \rho$ (the maximum and minimum curvatures, respectively).

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- The number m is the number of periods of the curvature contained in one period of γ_d. (Number of lobes).

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Let n and m be two relatively prime natural numbers satisfying $m < 2n < \sqrt{2}\, m$. Then, there exists a unique closed critical curve for $\mathbf{\Theta}$.

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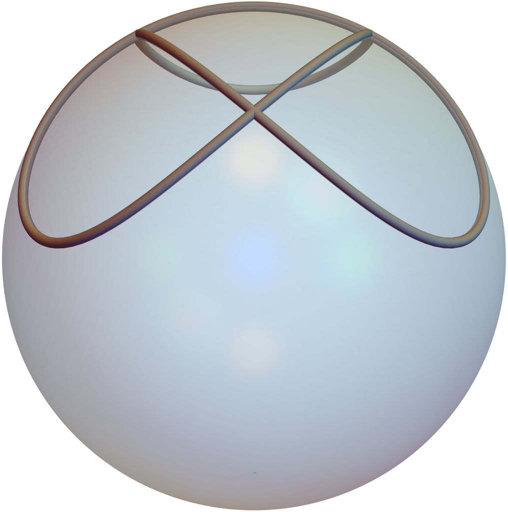
- In particular, there are no closed and simple critical curves.
- The "simplest" possible choice is $\gamma_{2,3}$.

Illustrations



$\gamma_{2,3}$

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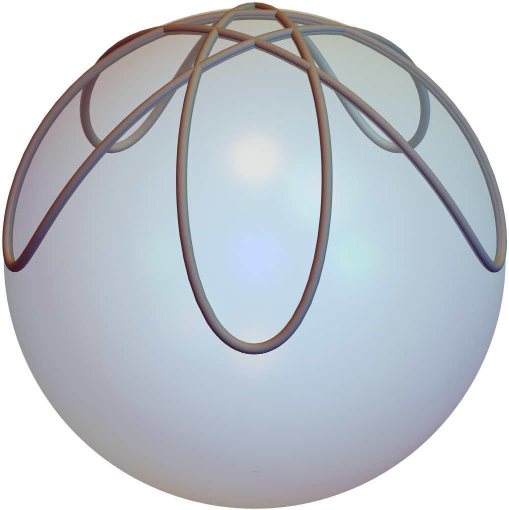

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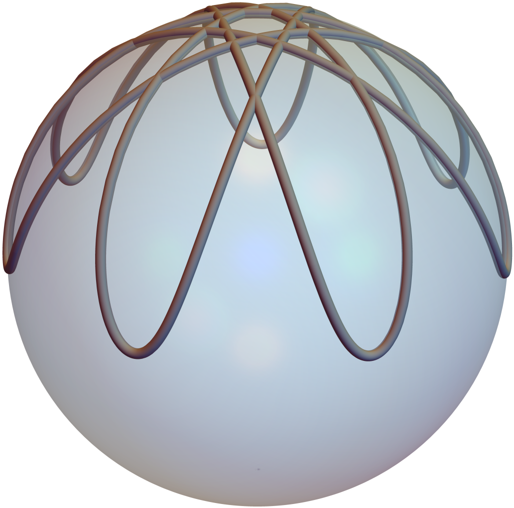

$\gamma_{4,7}$

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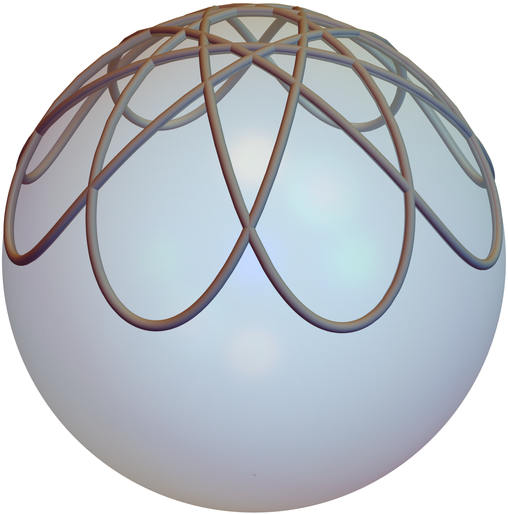

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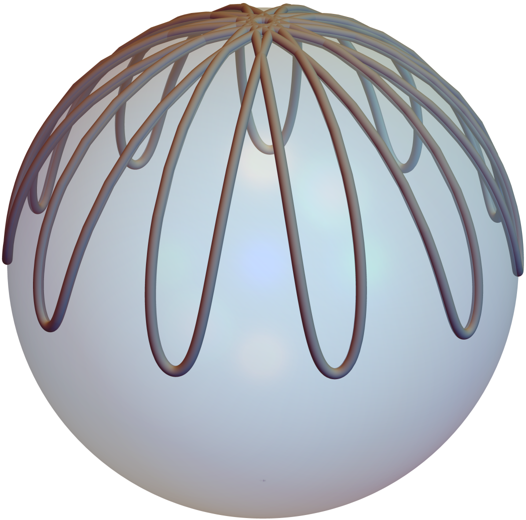

$\gamma_{6,11}$

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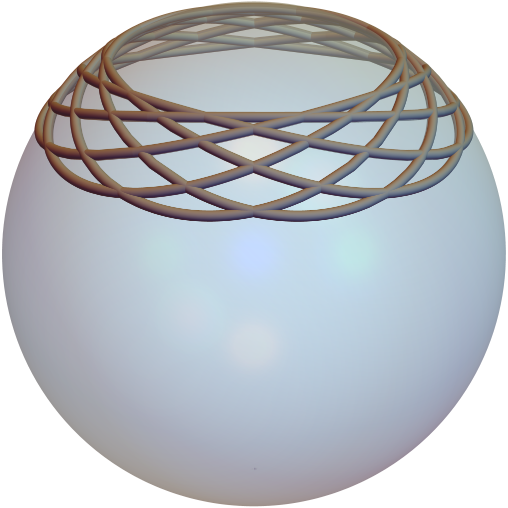

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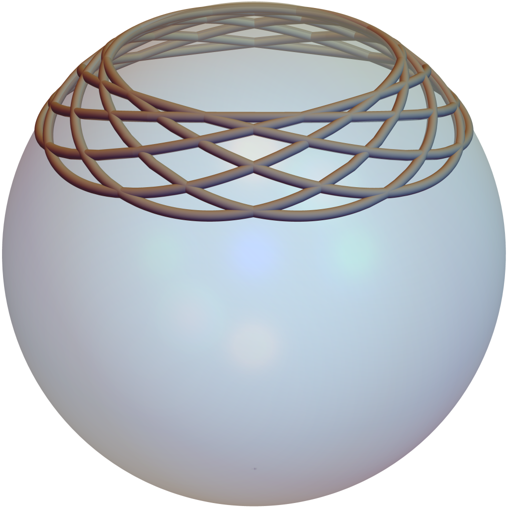

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Proposition (Langer & Singer, 1984)

Consider $M^2(\rho)$ embedded as a totally geodesic surface of $M^3(\rho)$. Then, the vector field

$$\mathcal{I} = \frac{1}{2\sqrt{\kappa}} B$$

is a Killing vector field along critical curves.

Binormal Evolution Surfaces

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3. We construct the binormal evolution surface (Garay & P., 2016)

$$S_\gamma = \{\phi_t(\gamma(s))\}\,.$$

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S_γ is either a flat isoparametric surface (when κ is constant), or it is a rotational surface (when κ is nonconstant). In particular, if $d > 0$ it is a spherical rotational surface.

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2. Since γ is critical for $\mathbf{\Theta}$,

Theorem (Arroyo, Garay & P., 2018)

S_γ is a minimal surface.

Characterization of (Rotational) Minimal Surfaces

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Any rotational minimal surface $S \subset M^3(\rho)$ is, locally, either a ruled surface or it is spanned by a planar critical curve for

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- We proved something more general, namely, any CMC ξ-invariant surface is, locally, spanned by a critical curve of an extension of $\mathbf{\Theta}$.

Other Applications of the Theory

Consider the 2-dimensional analogue of the Blaschke's variational problem, namely,

$$\mathcal{W}(\Sigma) := \int_\Sigma \sqrt{H}\, dA\,,$$

acting on the space of smooth weakly convex ($H > 0$ and $K \geq 0$) immersions.

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Euler-Lagrange Equation

$$2\sqrt{H}\,\Delta\left(\frac{1}{\sqrt{H}}\right) - 4H^2 + 2(2\rho - K) = 0\,.$$

Existence of Critical Tori

Theorem (P., 2020)

The preimage of a closed curve γ through the standard Hopf mapping $\mathbb{S}^2 \to \mathbb{S}^3$ is a critical torus for $\mathcal{W}$ if and only if γ is critical for $\mathbf{\Theta}$.

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- For every pair of relatively prime natural numbers (n, m) satisfying $m < 2n < \sqrt{2}\, m$, there exists a unique Hopf tori critical for $\mathcal{W}$.
- None of them is embedded.
- All are unstable (Gruber, Toda & P., Preprint).

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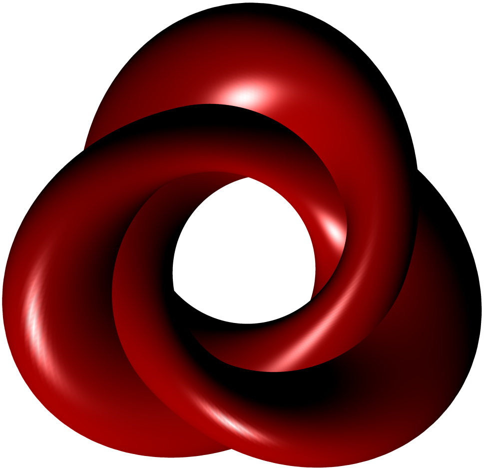

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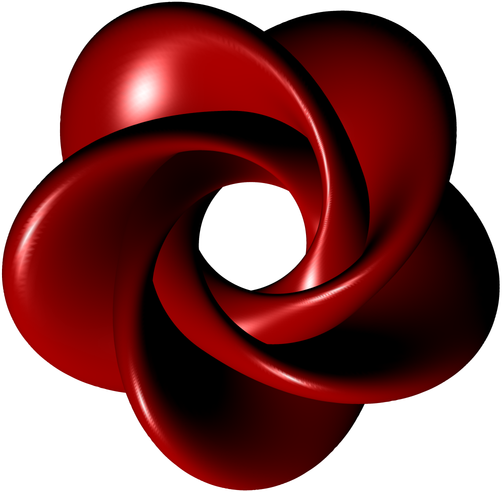

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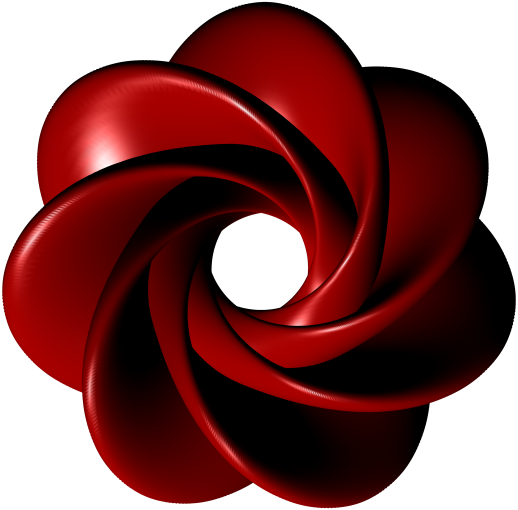

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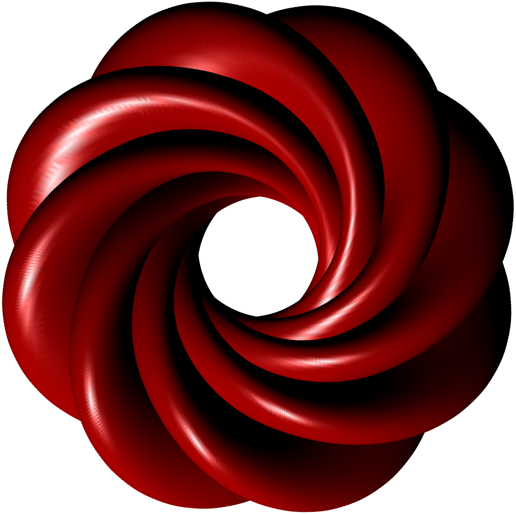

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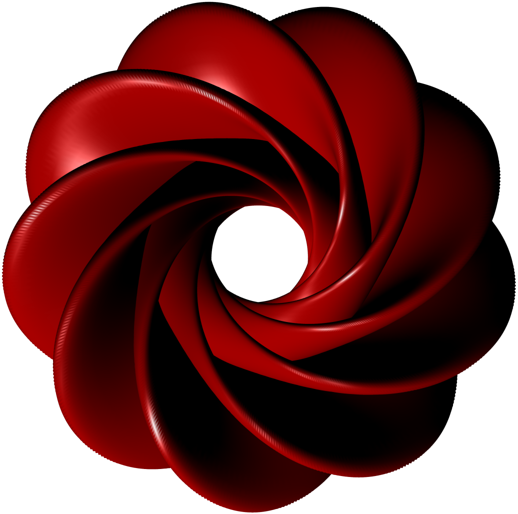
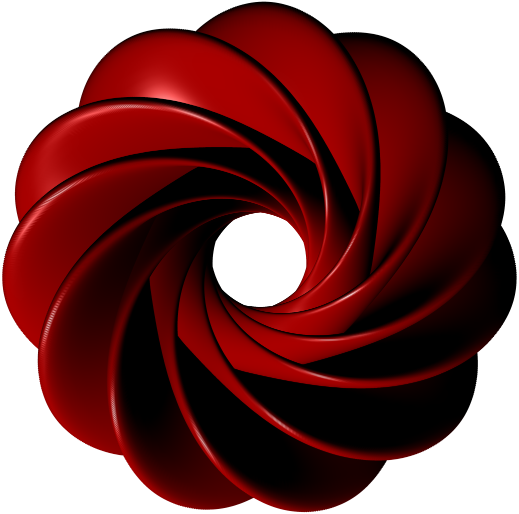


Illustrations



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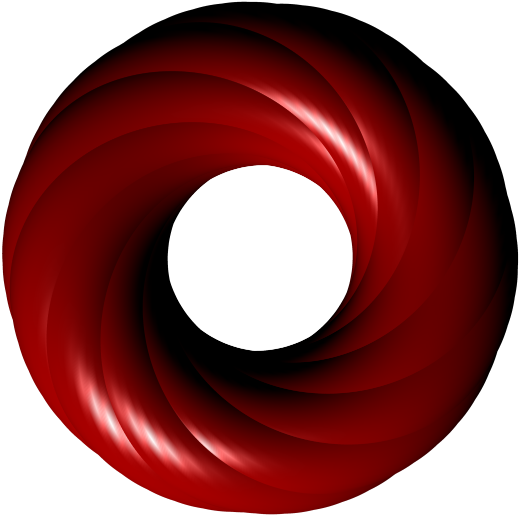

THE END

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Thank You!