Construction of Rotational Constant Skew Curvature Surfaces in Space Forms

Álvaro Pámpano Llarena

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From now on we will discard these cases.

Applications and Previous Studies

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This presentation is based on:

- R. López and —, Classification of rotational surfaces with constant skew curvature in 3-space forms, *J. Math. Anal. Appl.* **489** (2020), 124195.

Exponential Type Curvature Energy

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For any non-zero real constant μ, we consider the exponential type curvature energy

$$\boldsymbol{\Theta}_\mu(\gamma) := \int_\gamma e^{\mu\kappa} = \int_0^L e^{\mu\kappa(s)}\, ds$$

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Euler-Lagrange equation

Regardless of the boundary conditions, any critical curve for $\mathbf{\Theta}_\mu$ must satisfy

$$\frac{d^2}{ds^2}\left(e^{\mu\kappa}\right) + \left(\kappa^2 - \frac{\kappa}{\mu} + \rho\right) e^{\mu\kappa} = 0\,.$$

We will call them, simply, critical curves.

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3. If the critical curve has non constant curvature, then

$$\mu^4 \kappa_s^2 = d e^{-2\mu\kappa} - (\mu\kappa - 1)^2 - \rho\mu^2$$

for $d \in \mathbb{R}$ represents a first integral of the Euler-Lagrange equation.

Killing Vector Fields Along Curves

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- Killing vector fields along γ can be extended to Killing vector fields on the whole $M^3(\rho)$. The extension is unique.

Binormal Evolution Surfaces

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Let $\gamma(s) \subset M^2(\rho)$ be any critical curve for $\mathbf{\Theta}_\mu$. (We consider $M^2(\rho) \subset M^3(\rho)$ and γ being planar, i.e. $\tau = 0$.)

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1. Consider the Killing vector field along γ in the direction of the (constant) binormal vector field:

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3. Since $M^3(\rho)$ is complete, the one-parameter group of isometries determined by ξ is $\{\phi_t\,,\, t \in \mathbb{R}\}$.
4. We construct the binormal evolution surface **(Garay & —, 2016)**

$$S_\gamma := \{x(s,t) := \phi_t\,(\gamma(s))\}\,.$$

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Theorem (Arroyo, Garay & —, 2017)

The binormal evolution surface S_γ is either a flat isoparametric surface (when $\kappa(s) = \kappa_o$ is constant); or, it is a rotational surface (when $\kappa(s)$ is not constant).

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- Since $\gamma(s)$ is a critical curve for $\mathbf{\Theta}_\mu$,

Theorem (López & —, 2020)

The binormal evolution surface S_γ is a constant skew curvature surface. It verifies:

$$\kappa_1 = \kappa_2 + c\,, \qquad (\kappa_i \text{ principal curvatures})$$

for $c = 1/\mu$.

Characterization of Profile Curves

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Let $S \subset M^3(\rho)$ be a (non-isoparametric) rotational surface with constant skew curvature. If γ is a profile curve of S, then the curvature κ of γ satisfies the Euler-Lagrange equation associated to the exponential type curvature energy

$$\mathbf{\Theta}_\mu(\gamma) = \int_\gamma e^{\mu\kappa}$$

where $\mu = 1/c$.

Profile Curves in $\mathbb{R}^2$

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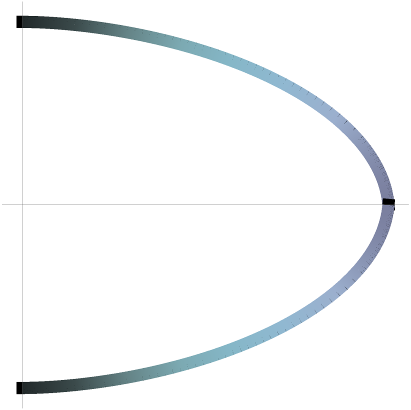

FIGURE: Oval Type Critical Curve

Profile Curves in $\mathbb{R}^2$

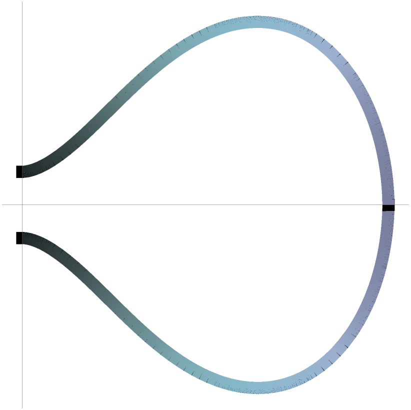

FIGURE: Simple Biconcave Type Critical Curve

Profile Curves in $\mathbb{R}^2$

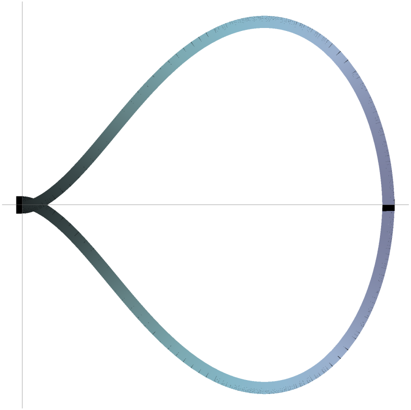

FIGURE: Figure-Eight Type Critical Curve

Profile Curves in $\mathbb{R}^2$

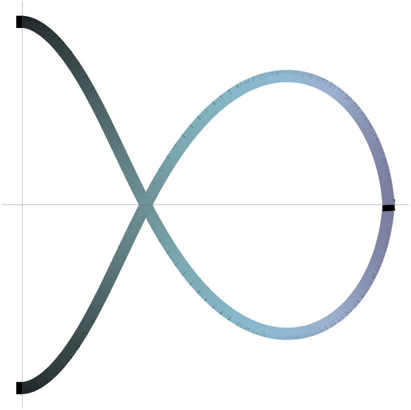

FIGURE: Non-Simple Biconcave Type Critical Curve

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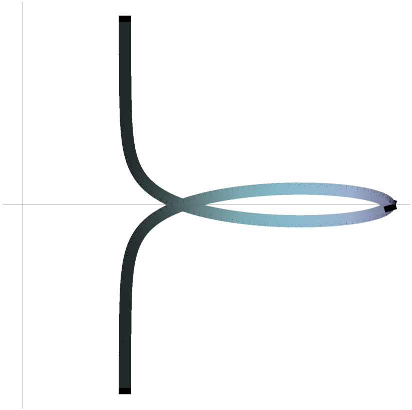

FIGURE: Borderline Type Critical Curve

Profile Curves in $\mathbb{R}^2$

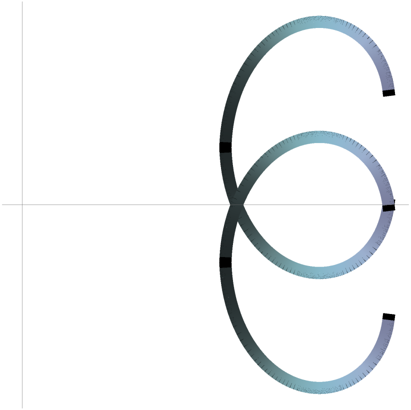

FIGURE: Orbit-Like Type Critical Curve

Profile Curves in $\mathbb{S}^2(\rho)$

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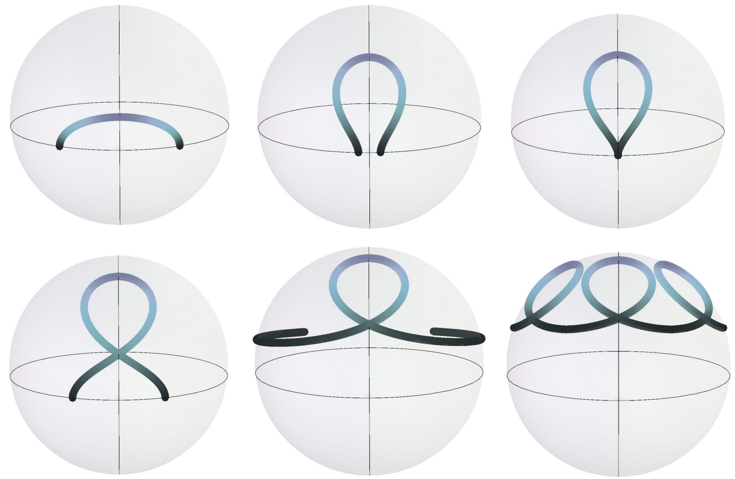

Profile Curves in $\mathbb{H}^2(\rho)$

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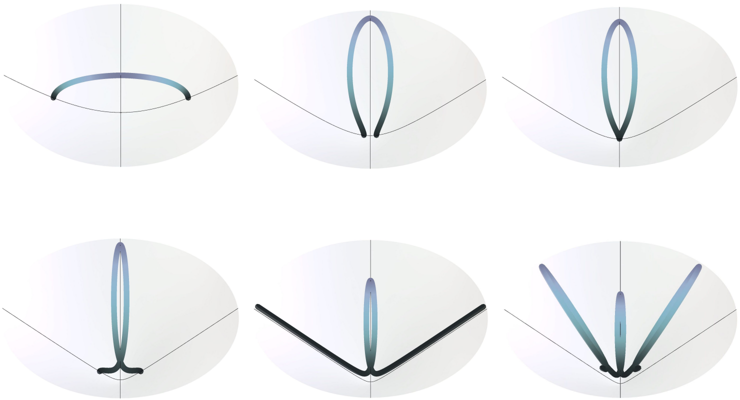

THE END

Thank You!