Generalized Elastic Curves in the Sphere $\mathbb{S}^2$ and their Applications

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Historical Background

In 1691, J. Bernoulli formulated the problem of quantifying the bending deformation of rods. According to a general principle originally due to D. Bernoulli, an elastic rod should bend along the curve which minimizes the potential energy of the strain under suitable constraints. Following this idea, L. Euler classified in 1744 all the possible qualitative types for rods in untwisted planar configurations. This problem was, in fact, posed more generally by D. Bernoulli, who proposed in a letter of 1738 to L. Euler to investigate extrema of what is now referred to as the p-Bernoulli's bending functionals. Since then, several particular cases other than the classical $p = 2$ have received widespread attention. For instance, in 1921-1923, W. Blaschke considered the cases $p = 1/2$ and $p = 1/3$ in $\mathbb{R}^3$.

Variational Problem

For every $p \in \mathbb{R}$, consider the **Bernoulli's bending functionals** defined by

$$\Theta_p(\gamma) := \int_\gamma \kappa^p \, ds \,,$$

where κ is the curvature of the **spherical** curve $\gamma : I \subseteq \mathbb{R} \longrightarrow \mathbb{S}^2$ parameterized by the arc length $s \in I$. Its associated Euler-Lagrange equation is

$$p \frac{d^2}{ds^2}\left(\kappa^{p-1}\right) + (p-1)\,\kappa^{p+1} + p\,\kappa^{p-1} = 0\,. \tag{1}$$

Definition. *Those curves whose curvature κ is identically zero or a solution of (1) are called* ***generalized elastic curves****.*

One of the main interests of the theory is to find (non-circular) closed generalized elastic curves. The case $p = 2$ was studied by J. Langer and D. A. Singer in 1984. They proved that there exists a bi-parametric family of closed non-circular elastic curves, including simple ones. With contributions from several papers, the following result was obtained for other values of $p \in \mathbb{R}$:

Theorem ([2, 3, 4])

Let γ be a (non-circular) closed generalized elastic curve. Then, either $p = 2$ or $p \in (0, 1)$. Furthermore, for every $p \in (0, 1)$ and any pair of relatively prime natural numbers satisfying $m < 2n < \sqrt{2}\,m$, there exists a non-circular closed generalized elastic curve.

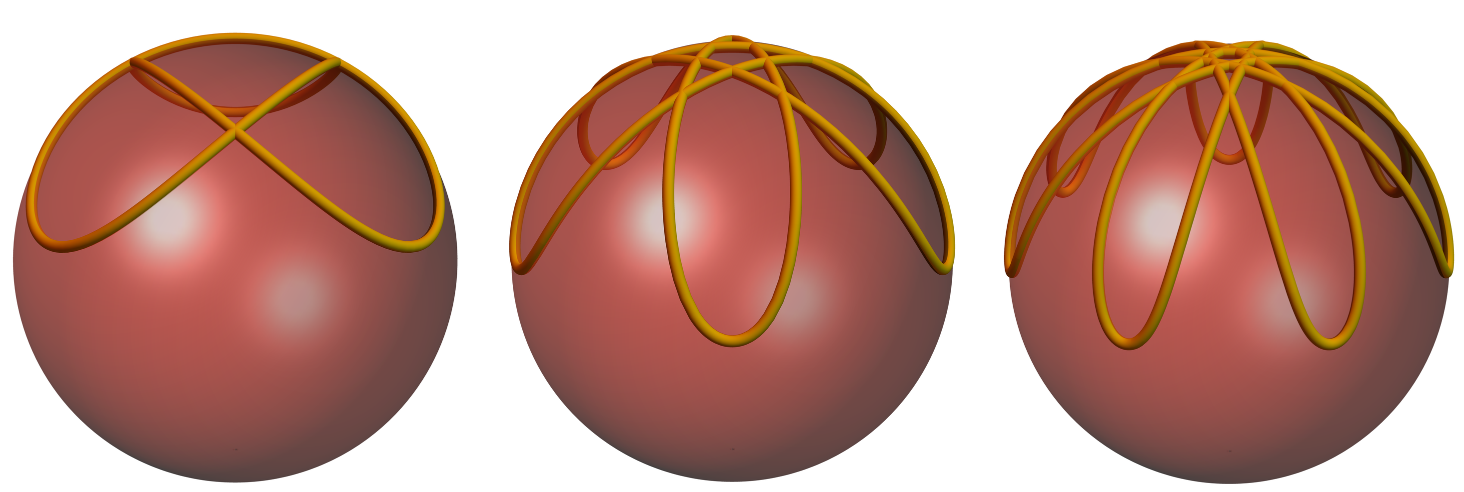

Figure. *Three closed spherical generalized elastic curves for $p = 1/2$.*

The number n represents the number of times the curve winds around the pole, i.e., it is the winding number, while m is the number of periods of the curvature needed to close the curve, i.e., the number of lobes.

Rotational Hypersurfaces

Minimal Surfaces in $\mathbb{S}^3$

Definition. *A surface immersed in $\mathbb{S}^3$ is a* ***minimal surface*** *if its mean curvature H is identically zero.*

Minimal surfaces in $\mathbb{S}^3$ have played a major role in Mathematics in the last decades. In 1966, F. J. Almgren proved that any immersed minimal topological sphere must be congruent to the equator. A few years later, in 1970, H. B. Lawson conjectured that all embedded minimal tori are congruent to the Clifford torus. This was proven in 2013 by S. Brendle. Focusing on rotational surfaces:

Theorem ([1, 2])

A rotational surface immersed in $\mathbb{S}^3$ is minimal if and only if its profile curve is a generalized elastic curve for $p = 1/2$.

From the results of closed generalized elastic curves for $p = 1/2$ it can be shown, in a way different from the classical result, that apart from the Clifford torus:

Corollary ([2])

For any pair of relatively prime natural numbers satisfying $m < 2n < \sqrt{2}\,m$, there exists a unique (up to congruence) rotational minimal torus immersed in $\mathbb{S}^3$.

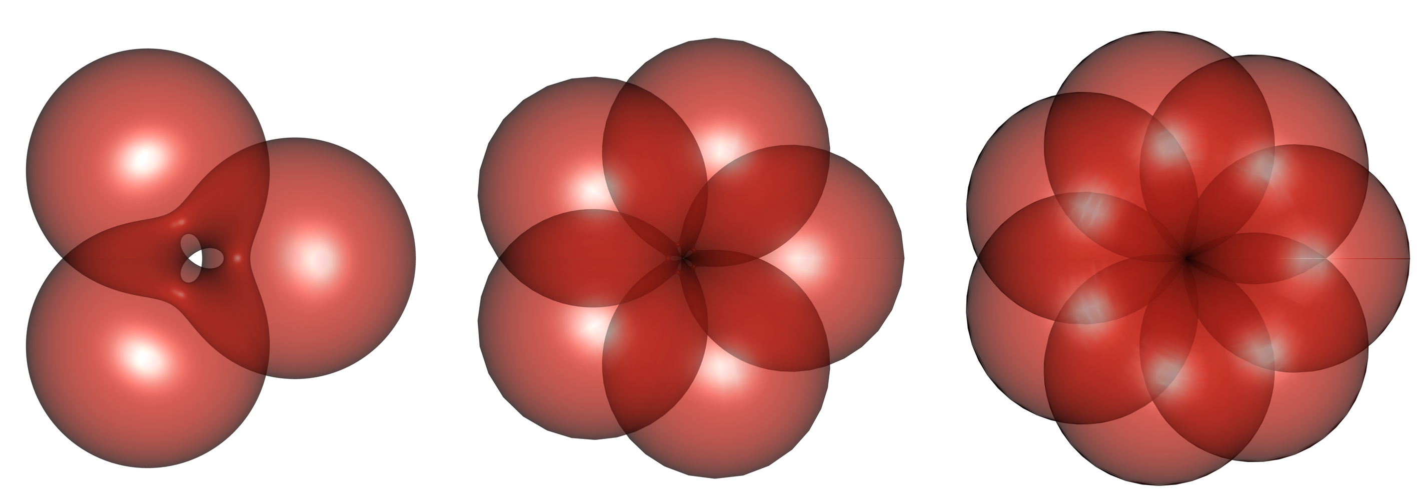

Figure. *Stereographic projection of three rotational minimal tori in $\mathbb{S}^3$.*

Biconservative Hypersurfaces in $\mathbb{S}^r$

Definition. *A hypersurface immersed in $\mathbb{S}^r$ ($r \geq 3$) is* ***biconservative*** *if*

$$2S_\eta\left(\mathrm{grad} H\right) + (r-1) H \,\mathrm{grad} H = 0\,, \tag{2}$$

where S_η is the shape operator and H is the mean curvature function.

Following the description of D. Hilbert in 1924, biconservative hypersurfaces can be interpreted as those hypersurfaces whose stress-energy tensor associated with the bienergy is conservative. Constant mean curvature (CMC) hypersurfaces satisfy (2). Thus, the main interest is to study non-CMC biconservative hypersurfaces for which:

Theorem ([4])

A non-CMC rotational hypersurface immersed in $\mathbb{S}^r$ is biconservative if and only if its profile curve is a non-circular generalized elastic curve for $p = (r-2)/(r+1)$.

An open problem of the theory was to determine if closed non-CMC biconservative hypersurfaces could exist. The question was answered affirmatively for every dimension:

Corollary ([4])

For every $r \geq 3$ and any pair of relatively prime natural numbers satisfying $m < 2n < \sqrt{2}\,m$, there exists a closed non-CMC biconservative hypersurface immersed in $\mathbb{S}^r$.

Hopf Tori

Definition. *A closed surface immersed in $\mathbb{S}^3$ is a* ***generalized Willmore surface*** *if it is a critical point of the functional*

$$\mathcal{W}_p(X) := \int_\Sigma H^p \, dA\,,$$

where H is the mean curvature function.

Employing the Symmetric Criticality Principle of R. Palais and the Hopf tori constructed by U. Pinkall, non-trivial generalized Willmore surfaces were obtained:

Theorem ([3, 5])

For every $p \in (0, 1)$ and any pair of relatively prime natural numbers satisfying $m < 2n < \sqrt{2}\,m$, there exists a (non-trivial) generalized Willmore Hopf tori immersed in $\mathbb{S}^3$.

The case $p = 2$, studied by U. Pinkall in 1985, gave rise to a family of non-conformally minimal Willmore tori.

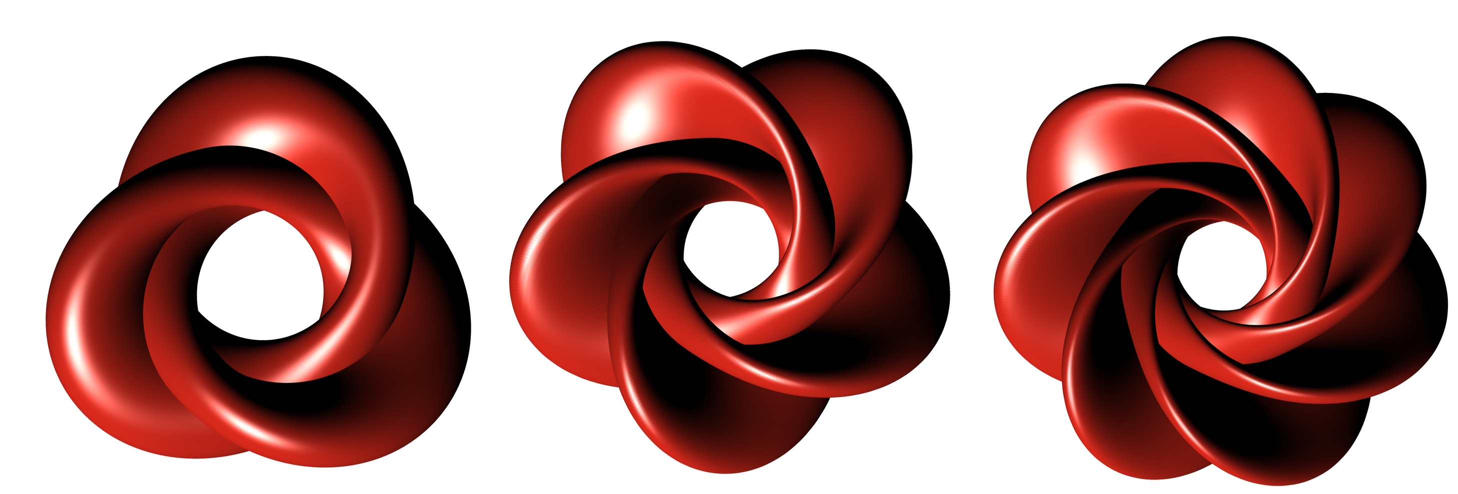

Figure. *Stereographic projection of three (non-trivial) generalized Willmore Hopf tori in $\mathbb{S}^3$ for $p = 1/2$.*

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