eman ta zabal zazu
Universidad del País Vasco
Euskal Herriko Unibertsitatea

ZTF-FCT
Zientzia eta Teknologia Fakultatea
Facultad de Ciencia y Tecnología

On Extremals of Curvature Energies used in Visual Curve Completion

Álvaro Pámpano Llarena

22nd International Summer School on Global Analysis and its Applications

Krakow, 21-25 August, 2017

Objectives

Objectives

Objective 1

Introduce the unit tangent bundle $\mathbb{R}^2 \times \mathbb{S}^1$ model for the primary visual cortex ([1], [4] and [5]).

Objectives

Objective 1

Introduce the unit tangent bundle $\mathbb{R}^2 \times \mathbb{S}^1$ model for the primary visual cortex ([1], [4] and [5]).

Objective 2

Compare extremals of different curvature energies that are used in visual curve completion [1].

Objectives

Objective 1

Introduce the unit tangent bundle $\mathbb{R}^2 \times \mathbb{S}^1$ model for the primary visual cortex ([1], [4] and [5]).

Objective 2

Compare extremals of different curvature energies that are used in visual curve completion [1].

- Length (equivalently, total curvature type energy [2])
- Elastic Energy
- Total Squared Torsion

Index

Index

Index

Index

1. Motivation
2. Total Curvature Type Energy
3. Other Curvature Energy Functionals

Motivation

1. Primary Visual Cortex V1

Motivation

1. Primary Visual Cortex V1
2. Sub-Riemannian Structure of the Visual Cortex

Motivation

1. Primary Visual Cortex V1
2. Sub-Riemannian Structure of the Visual Cortex
3. Gradient-Descent Method

Primary Visual Cortex V1

Primary Visual Cortex V1

Unit Tangent Bundle ([1] and [4])

The unit tangent bundle of the plane, $\mathbb{R}^2 \times \mathbb{S}^1$, can be used as an abstraction to study the organization and mechanisms of V1.

Unit Tangent Bundle ([1] and [4])

The unit tangent bundle of the plane, $\mathbb{R}^2 \times \mathbb{S}^1$, can be used as an abstraction to study the organization and mechanisms of V1.

- Each point (x, y, θ) represents a column of cells associated with a point of retinal data $(x, y) \in \mathbb{R}^2$,

Unit Tangent Bundle ([1] and [4])

The unit tangent bundle of the plane, $\mathbb{R}^2 \times \mathbb{S}^1$, can be used as an abstraction to study the organization and mechanisms of V1.

- Each point (x, y, θ) represents a column of cells associated with a point of retinal data $(x, y) \in \mathbb{R}^2$, all of which are adjusted to the orientation given by the angle $\theta \in \mathbb{S}^1$.

Primary Visual Cortex V1

Unit Tangent Bundle ([1] and [4])

The unit tangent bundle of the plane, $\mathbb{R}^2 \times \mathbb{S}^1$, can be used as an abstraction to study the organization and mechanisms of V1.

- Each point (x, y, θ) represents a column of cells associated with a point of retinal data $(x, y) \in \mathbb{R}^2$, all of which are adjusted to the orientation given by the angle $\theta \in \mathbb{S}^1$.
- The vector $(\cos\theta, \sin\theta)$ is the direction of maximal rate of change of brightness at point (x, y) of the picture seen by the eye [5].

Primary Visual Cortex V1

Unit Tangent Bundle ([1] and [4])

The unit tangent bundle of the plane, $\mathbb{R}^2 \times \mathbb{S}^1$, can be used as an abstraction to study the organization and mechanisms of V1.

- Each point (x, y, θ) represents a column of cells associated with a point of retinal data $(x, y) \in \mathbb{R}^2$, all of which are adjusted to the orientation given by the angle $\theta \in \mathbb{S}^1$.
- The vector $(\cos\theta, \sin\theta)$ is the direction of maximal rate of change of brightness at point (x, y) of the picture seen by the eye [5].
- When the cortex cells are stimulated by an image,

Unit Tangent Bundle ([1] and [4])

The unit tangent bundle of the plane, $\mathbb{R}^2 \times \mathbb{S}^1$, can be used as an abstraction to study the organization and mechanisms of V1.

- Each point (x, y, θ) represents a column of cells associated with a point of retinal data $(x, y) \in \mathbb{R}^2$, all of which are adjusted to the orientation given by the angle $\theta \in \mathbb{S}^1$.
- The vector $(\cos\theta, \sin\theta)$ is the direction of maximal rate of change of brightness at point (x, y) of the picture seen by the eye [5].
- When the cortex cells are stimulated by an image, the border of the image gives a curve inside the space $\mathbb{R}^2 \times \mathbb{S}^1$,

Primary Visual Cortex V1

Unit Tangent Bundle ([1] and [4])

The unit tangent bundle of the plane, $\mathbb{R}^2 \times \mathbb{S}^1$, can be used as an abstraction to study the organization and mechanisms of V1.

- Each point (x, y, θ) represents a column of cells associated with a point of retinal data $(x, y) \in \mathbb{R}^2$, all of which are adjusted to the orientation given by the angle $\theta \in \mathbb{S}^1$.
- The vector $(\cos\theta, \sin\theta)$ is the direction of maximal rate of change of brightness at point (x, y) of the picture seen by the eye [5].
- When the cortex cells are stimulated by an image, the border of the image gives a curve inside the space $\mathbb{R}^2 \times \mathbb{S}^1$, but restricted to be tangent to a specific distribution.

Sub-Riemannian Structure of $\mathbb{R}^2 \times \mathbb{S}^1$

We consider the space $\mathbb{R}^2 \times \mathbb{S}^1$.

Sub-Riemannian Structure of $\mathbb{R}^2 \times \mathbb{S}^1$

We consider the space $\mathbb{R}^2 \times \mathbb{S}^1$.

- Take the distribution $\mathcal{D} = Ker(\sin\theta dx - \cos\theta dy)$.

Sub-Riemannian Structure of $\mathbb{R}^2 \times \mathbb{S}^1$

We consider the space $\mathbb{R}^2 \times \mathbb{S}^1$.

- Take the distribution $\mathcal{D} = Ker(\sin\theta dx - \cos\theta dy)$.
- The distribution $\mathcal{D}$ is spanned by

$$X_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\,, \quad X_2 = \frac{\partial}{\partial \theta}$$

Sub-Riemannian Structure of $\mathbb{R}^2 \times \mathbb{S}^1$

We consider the space $\mathbb{R}^2 \times \mathbb{S}^1$.

- Take the distribution $\mathcal{D} = Ker(\sin\theta dx - \cos\theta dy)$.
- The distribution $\mathcal{D}$ is spanned by

$$X_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial \theta}$$

- The distribution $\mathcal{D}$ is bracket-generating.

Sub-Riemannian Structure of $\mathbb{R}^2 \times \mathbb{S}^1$

We consider the space $\mathbb{R}^2 \times \mathbb{S}^1$.

- Take the distribution $\mathcal{D} = Ker(\sin\theta dx - \cos\theta dy)$.
- The distribution $\mathcal{D}$ is spanned by

$$X_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial \theta}$$

- The distribution $\mathcal{D}$ is bracket-generating.
- Finally, with the inner product $\langle , \rangle$ defined by making X_1 and X_2 everywhere orthonormal, we obtain

Sub-Riemannian Structure of $\mathbb{R}^2 \times \mathbb{S}^1$

We consider the space $\mathbb{R}^2 \times \mathbb{S}^1$.

- Take the distribution $\mathcal{D} = Ker(\sin\theta dx - \cos\theta dy)$.
- The distribution $\mathcal{D}$ is spanned by

$$X_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial \theta}$$

- The distribution $\mathcal{D}$ is bracket-generating.
- Finally, with the inner product $\langle , \rangle$ defined by making X_1 and X_2 everywhere orthonormal, we obtain

Sub-Riemannian Structure of V1

The unit tangent bundle $\mathbb{R}^2 \times \mathbb{S}^1$ is a 3-dimensional sub-Riemannian manifold $(\mathbb{R}^2 \times \mathbb{S}^1, \mathcal{D}, \langle , \rangle)$.

Gradient-Descent Method

If a piece of the contour of a picture is missing to the eye vision (or maybe it is covered by an object),

Gradient-Descent Method

If a piece of the contour of a picture is missing to the eye vision (or maybe it is covered by an object), then the brain tends to complete the curve by minimizing some kind of energy

Gradient-Descent Method

If a piece of the contour of a picture is missing to the eye vision (or maybe it is covered by an object), then the brain tends to complete the curve by minimizing some kind of energy (length, elastic energy, total squared torsion,...).

Gradient-Descent Method

If a piece of the contour of a picture is missing to the eye vision (or maybe it is covered by an object), then the brain tends to complete the curve by minimizing some kind of energy (length, elastic energy, total squared torsion,...).

Problem

In general, the boundary value problem is very hard to solve analytically.

Gradient-Descent Method

If a piece of the contour of a picture is missing to the eye vision (or maybe it is covered by an object), then the brain tends to complete the curve by minimizing some kind of energy (length, elastic energy, total squared torsion,...).

Problem

In general, the boundary value problem is very hard to solve analytically.

To overcome this difficulty, our group has developed a numerical approach,

Gradient-Descent Method

If a piece of the contour of a picture is missing to the eye vision (or maybe it is covered by an object), then the brain tends to complete the curve by minimizing some kind of energy (length, elastic energy, total squared torsion,...).

Problem

In general, the boundary value problem is very hard to solve analytically.

To overcome this difficulty, our group has developed a numerical approach,

XEL-platform [3] (www.ikergeometry.org)

A gradient descent method useful for an ample family of functionals defined on certain spaces of curves satisfying both affine and isoperimetric constraints.

Total Curvature Type Energy

Total Curvature Type Energy

1. Sub-Riemannian Geodesics

Total Curvature Type Energy

1. Sub-Riemannian Geodesics
2. $\mathcal{F}(\gamma) = \int_\gamma \sqrt{\kappa^2(s) + a^2}\, ds$ in $\mathbb{R}^2$

Sub-Riemannian Geodesics

- A $\mathcal{D}$-curve on M^3 is a curve which is always tangent to $\mathcal{D}$.

Sub-Riemannian Geodesics

- A $\mathcal{D}$-curve on M^3 is a curve which is always tangent to $\mathcal{D}$.
- Every $\mathcal{D}$-curve $\gamma(t) = (x(t), y(t), \theta(t))$ is the lift of a regular curve $\alpha(t)$ in $\mathbb{R}^2$ if $\gamma^*(\cos\theta dx + \sin\theta dy) \neq 0$.

Sub-Riemannian Geodesics

- A $\mathcal{D}$-curve on M^3 is a curve which is always tangent to $\mathcal{D}$.
- Every $\mathcal{D}$-curve $\gamma(t) = (x(t), y(t), \theta(t))$ is the lift of a regular curve $\alpha(t)$ in $\mathbb{R}^2$ if $\gamma^*(\cos\theta dx + \sin\theta dy) \neq 0$.
- Conversely, every regular curve $\alpha(t)$ in the plane may be lifted to a $\mathcal{D}$-curve $\gamma(t)$ by setting $\theta(t)$ equal to the angle between $\alpha'(t)$ and the x-axis.

Sub-Riemannian Geodesics

- A $\mathcal{D}$-curve on M^3 is a curve which is always tangent to $\mathcal{D}$.
- Every $\mathcal{D}$-curve $\gamma(t) = (x(t), y(t), \theta(t))$ is the lift of a regular curve $\alpha(t)$ in $\mathbb{R}^2$ if $\gamma^*(\cos\theta dx + \sin\theta dy) \neq 0$.
- Conversely, every regular curve $\alpha(t)$ in the plane may be lifted to a $\mathcal{D}$-curve $\gamma(t)$ by setting $\theta(t)$ equal to the angle between $\alpha'(t)$ and the x-axis.

Relation with Total Curvature Type Energies ([1], [2] and [4])

Geodesics in V1 are obtained by lifting to $M^3 = \mathbb{R}^2 \times \mathbb{S}^1$ minimizers in $\mathbb{R}^2$ of

Sub-Riemannian Geodesics

- A $\mathcal{D}$-curve on M^3 is a curve which is always tangent to $\mathcal{D}$.
- Every $\mathcal{D}$-curve $\gamma(t) = (x(t), y(t), \theta(t))$ is the lift of a regular curve $\alpha(t)$ in $\mathbb{R}^2$ if $\gamma^*(\cos\theta dx + \sin\theta dy) \neq 0$.
- Conversely, every regular curve $\alpha(t)$ in the plane may be lifted to a $\mathcal{D}$-curve $\gamma(t)$ by setting $\theta(t)$ equal to the angle between $\alpha'(t)$ and the x-axis.

Relation with Total Curvature Type Energies ([1], [2] and [4])

Geodesics in V1 are obtained by lifting to $M^3 = \mathbb{R}^2 \times \mathbb{S}^1$ minimizers in $\mathbb{R}^2$ of

$$\mathcal{F}(\alpha) = \int_\alpha \sqrt{1 + \kappa^2(s)}\, ds\,.$$

$\mathcal{F}(\alpha) = \int_\alpha \sqrt{\kappa^2(s) + a^2}\, ds$ IN $\mathbb{R}^2$

By the hypercolumnar organization of the visual cortex,

$\mathcal{F}(\alpha) = \int_\alpha \sqrt{\kappa^2(s) + a^2}\, ds$ IN $\mathbb{R}^2$

By the hypercolumnar organization of the visual cortex, we may consider the functional

$$\mathcal{F}(\alpha) = \int_\alpha \sqrt{\kappa^2(s) + a^2}\, ds$$

acting on planar curves.

By the hypercolumnar organization of the visual cortex, we may consider the functional

$$\mathcal{F}(\alpha) = \int_\alpha \sqrt{\kappa^2(s) + a^2}\, ds$$

acting on planar curves.

Euler-Lagrange Equation ([1] and [2])

$$\frac{d^2}{ds^2}\left(\frac{\kappa}{\sqrt{\kappa^2 + a^2}}\right) - \frac{a^2\kappa}{\sqrt{\kappa^2 + a^2}} = 0$$

$\mathcal{F}(\alpha) = \int_\alpha \sqrt{\kappa^2(s) + a^2}\, ds$ IN $\mathbb{R}^2$

By the hypercolumnar organization of the visual cortex, we may consider the functional

$$\mathcal{F}(\alpha) = \int_\alpha \sqrt{\kappa^2(s) + a^2}\, ds$$

acting on planar curves.

Euler-Lagrange Equation ([1] and [2])

$$\frac{d^2}{ds^2}\left(\frac{\kappa}{\sqrt{\kappa^2 + a^2}}\right) - \frac{a^2\kappa}{\sqrt{\kappa^2 + a^2}} = 0$$

- If $a = 0$ we get the Total Curvature Functional,

$\mathcal{F}(\alpha) = \int_\alpha \sqrt{\kappa^2(s) + a^2}\, ds$ in $\mathbb{R}^2$

By the hypercolumnar organization of the visual cortex, we may consider the functional

$$\mathcal{F}(\alpha) = \int_\alpha \sqrt{\kappa^2(s) + a^2}\, ds$$

acting on planar curves.

Euler-Lagrange Equation ([1] and [2])

$$\frac{d^2}{ds^2}\left(\frac{\kappa}{\sqrt{\kappa^2 + a^2}}\right) - \frac{a^2\kappa}{\sqrt{\kappa^2 + a^2}} = 0$$

- If $a = 0$ we get the Total Curvature Functional, and therefore we know that any α is critical for it.

Solution of Euler-Lagrange Equation

If $a \neq 0$, we get the first integral of the Euler-Lagrange Equation,

Solution of Euler-Lagrange Equation

If $a \neq 0$, we get the first integral of the Euler-Lagrange Equation,

$$\kappa_s^2 = (\frac{\kappa^2 + a^2}{a^2})^2 (d\kappa^2 + a^2(d - a^2)).$$

Solution of Euler-Lagrange Equation

If $a \neq 0$, we get the first integral of the Euler-Lagrange Equation,

$$\kappa_s^2 = (\frac{\kappa^2 + a^2}{a^2})^2 (d\kappa^2 + a^2(d - a^2)).$$

Thus, we have that the curvature is given by,

$$\kappa(s) = \frac{a\sqrt{d - a^2} \tanh as}{\sqrt{a^2 - d \tanh^2 as}}.$$

Solution of Euler-Lagrange Equation

If $a \neq 0$, we get the first integral of the Euler-Lagrange Equation,

$$\kappa_s^2 = (\frac{\kappa^2 + a^2}{a^2})^2 (d\kappa^2 + a^2(d - a^2)).$$

Thus, we have that the curvature is given by,

$$\kappa(s) = \frac{a\sqrt{d - a^2}\tanh as}{\sqrt{a^2 - d\tanh^2 as}}.$$

And, therefore the critical curve α can be parametrized as,

$$\alpha(s) = (\int \cos \int \kappa, \int \sin \int \kappa).$$

Other Curvature Energy Functionals

1. Elastic Energy
2. Total Squared Torsion

Elasticae in $\mathbb{R}^2 \times \mathbb{S}^1$

Model of D. Munford (1992)

In order to reconstruct hidden contours, elasticae are the most probable curves.

Elasticae in $\mathbb{R}^2 \times \mathbb{S}^1$

Model of D. Munford (1992)

In order to reconstruct hidden contours, elasticae are the most probable curves.

Therefore, we will consider the elastic or bending energy

Elasticae in $\mathbb{R}^2 \times \mathbb{S}^1$

Model of D. Munford (1992)

In order to reconstruct hidden contours, elasticae are the most probable curves.

Therefore, we will consider the elastic or bending energy

$$\mathcal{F}(\gamma) = \int_\gamma \kappa^2(s)\, ds.$$

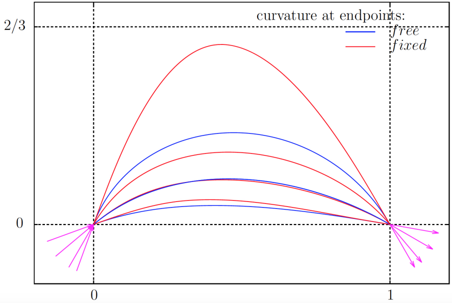
Elasticae in $\mathbb{R}^2 \times \mathbb{S}^1$

Model of D. Munford (1992)

In order to reconstruct hidden contours, elasticae are the most probable curves.

Therefore, we will consider the elastic or bending energy

$$\mathcal{F}(\gamma) = \int_\gamma \kappa^2(s)\, ds.$$



Total Squared Torsion

Another possibility in image reconstruction is to choose projections of minimizers of the total squared torsion in the unit tangent bundle $\mathbb{R}^2 \times \mathbb{S}^1$.

Total Squared Torsion

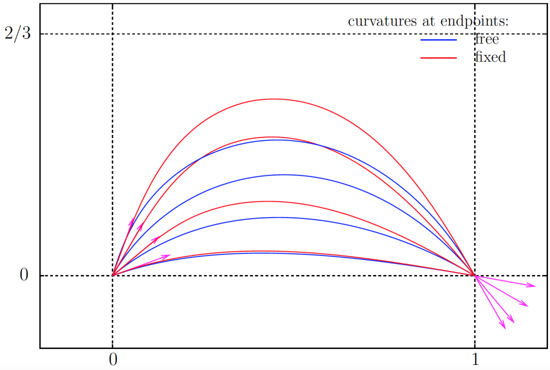
Another possibility in image reconstruction is to choose projections of minimizers of the total squared torsion in the unit tangent bundle $\mathbb{R}^2 \times \mathbb{S}^1$. This energy is defined by

$$\mathcal{F}(\gamma) = \int_\gamma \tau^2(s)\, ds.$$

Total Squared Torsion

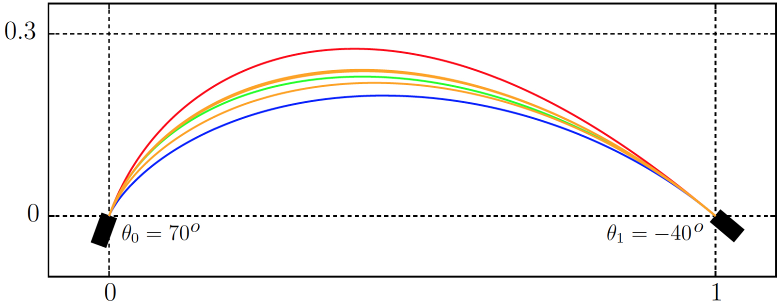
Another possibility in image reconstruction is to choose projections of minimizers of the total squared torsion in the unit tangent bundle $\mathbb{R}^2 \times \mathbb{S}^1$. This energy is defined by

$$\mathcal{F}(\gamma) = \int_\gamma \tau^2(s)\, ds.$$

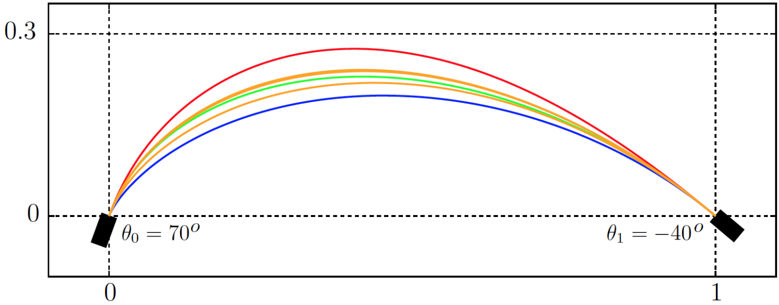


Comparison between Length, Bending Energy and Total Square Torsion

Comparison between Length, Bending Energy and Total Square Torsion

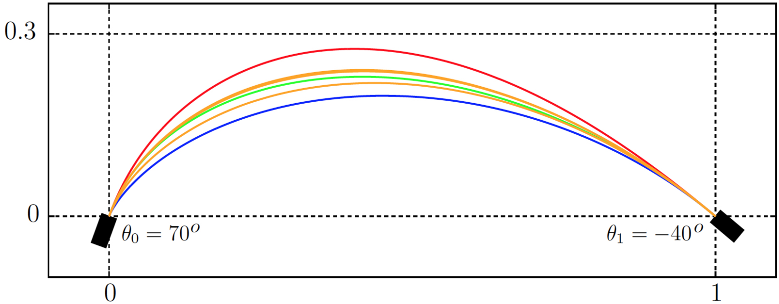


Comparison between Length, Bending Energy and Total Square Torsion



Length (Geodesic): Blue

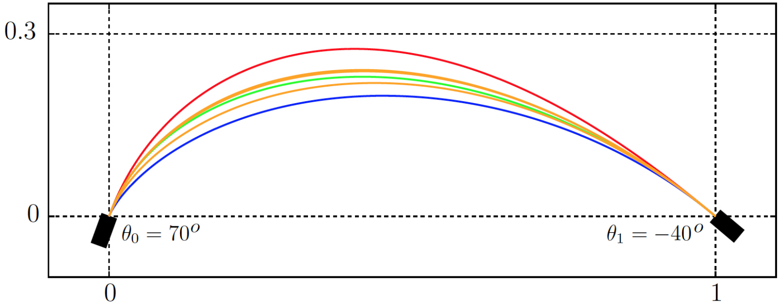
Comparison between Length, Bending Energy and Total Square Torsion



Length (Geodesic): Blue

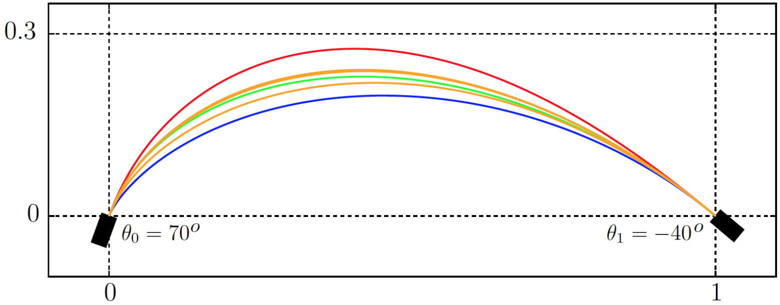
Elastic Energy: Green

Comparison between Length, Bending Energy and Total Square Torsion



Length (Geodesic): Blue
Elastic Energy: Green
Total Squared Torsion: Red (global minimun)

Comparison between Length, Bending Energy and Total Square Torsion



Length (Geodesic): Blue
Elastic Energy: Green
Total Squared Torsion: Red (global minimun) and Orange (local minima)

References

1. J. Arroyo, O. J. Garay and A. Pámpano, Curvature-Dependent Energies Minimizers and Visual Curve Completion, Non-Linear Dynamics, vol. 85 (2016), 1137-1156.
2. J. Arroyo, O. J. Garay and A. Pámpano, Extremal Curves of a Total Curvature Type Energy, Proc. 14th Int. Conf. Nonlinear Systems, Analysis and Chaos, (2015), 103-112.
3. J. Arroyo, O. J. Garay, J. J. Mencía and A. Pámpano, A Gradient-Descent Method for Langrangian Densities depending on Multiple Derivatives, In preparation.
4. G. Ben-Yosef and O. Ben-Shahar, A Tangent Bundle Theory for Visual Curve Completion, IEEE Trans. Pattern Anal. Mach. Intell., vol 34 (2012), 1263-1280.
5. R. Duits, U. Boscain, F. Rossi and Y. Sachkov, Association Fields via Cuspless Sub-Riemannian Geodesics in $SE(2)$, J. Math. Imaging Vis., vol 49 (2014), 384-417.

Thank You!

Acknowledgements: This research was supported by by MINECO-FEDER grant MTM2014-54804-P and Gobierno Vasco grant IT1094-16. The author has also been supported by Programa Predoctoral de Formacion de Personal Investigador No Doctor, Gobierno Vasco, 2015.