LECTURE NOTES

Math 2450, Calculus III with Applications

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1 Vectors in Plane and Space (Chapter 9)

Definition 1.1 *The plane can be described as the set,*

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\} ,$$

while the space is the set,

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\} .$$

Both, $\mathbb{R}^2$ and $\mathbb{R}^3$ are examples of vector and affine spaces.

Definition 1.2 *A vector $\vec{v}$ is an element of a vector space V. Roughly speaking, a vector is a mathematical object which has both:*

(i) Magnitude (norm), and

(ii) Direction.

(It can be represented by an arrow).

Remark 1.3 *A vector has an initial and final point. For instance, if the initial point is $P = (p_1, p_2, p_3)$ and the final point is $Q = (q_1, q_2, q_3)$, the vector $\vec{v} = \vec{PQ}$ is given by*

$$\vec{PQ} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle ,$$

i.e., "the components of $\vec{PQ}$ are the coordinates of Q minus the coordinates of P".
Vectors can be translated, so we can always think about them as fixed in the origin, that is, we may assume $P = (0, 0, 0)$.

Definition 1.4 *The norm of a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is given by*

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} .$$

(This quantity is always non-negative and, it is zero if and only if the vector $\vec{v}$ is identically zero).

Remark 1.5 *The norm of a vector comes from the Pythagorean Theorem.*

Proposition 1.6 (General Formula) *Let $\vec{v}$ be a vector with initial point $P = (p_1, p_2, p_3)$ and final point $Q = (q_1, q_2, q_3)$. Then, $\vec{v}$ can be expressed in terms of its components as*

$$\vec{v} = \vec{PQ} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle \,.$$

Moreover, the norm (magnitude) of $\vec{v}$ can be computed using

$$\|\vec{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + (q_3 - p_3)^2}\,.$$

Definition 1.7 *The* <u>*distance*</u> *between two points P and Q is defined as $d(P, Q) = \|\vec{PQ}\| = \|\vec{QP}\|$.*

Definition 1.8 *A* <u>*unit*</u> *vector is a vector of norm one. The* <u>*standard*</u> *(or,* <u>*canonical*</u>*) unit vectors are:*

$$\vec{i} = \langle 1, 0, 0 \rangle \,, \qquad \vec{j} = \langle 0, 1, 0 \rangle \,, \qquad \vec{k} = \langle 0, 0, 1 \rangle \,.$$

Remark 1.9 *Given a vector $\vec{v}$, there exists a unit vector with the same direction,*

$$\frac{\vec{v}}{\|\vec{v}\|}\,.$$

Proposition 1.10 *Operations:*

1. <u>*Scalar Multiplication*</u>*. Let $c > 0$ be a real number and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ a vector, then $c\vec{v}$ is a vector with the same direction as $\vec{v}$ and with norm $c\|\vec{v}\|$. If $c < 0$, the direction is the opposite and the norm is $|c|\|\vec{v}\|$. In terms of its components, for any $c \in \mathbb{R}$,*

$$c\vec{v} = \langle cv_1, cv_2, cv_3 \rangle \,.$$

2. <u>*Sum of Vectors*</u>*. Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{u} = \langle u_1, u_2, u_3 \rangle$. The sum $\vec{v} + \vec{u}$ is the diagonal of the parallelogram with sides $\vec{v}$ and $\vec{u}$. In terms of its components,*

$$\vec{v} + \vec{u} = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3 \rangle \,.$$

Remark 1.11 *The difference between two vectors $\vec{v}$ and $\vec{u}$ can be understood in terms of above two operations. Indeed,*

$$\vec{v} - \vec{u} = \vec{v} + (-\vec{u}) \,,$$

where $-\vec{u}$ is, precisely, the scalar multiplication of $\vec{u}$ by -1.

Notation 1.12 *Using above operations and the canonical vectors, a vector $\vec{v}$ can be denoted in the following two equivalent ways:*

$$\vec{v} = \langle v_1, v_2, v_3 \rangle = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}\,.$$

1.1 The Dot Product

Definition 1.13 *The* <u>*dot product*</u> *(or, also* <u>*inner*</u> *or* <u>*scalar*</u> *product) of two vectors* $\vec{u} = \langle u_1, u_2, u_3 \rangle$ *and* $\vec{v} = \langle v_1, v_2, v_3 \rangle$ *is the scalar (real number) given by*

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \, .$$

Proposition 1.14 (Properties of the Dot Product) *Let* $\vec{u}$, $\vec{v}$ *and* $\vec{w}$ *be three vectors and* $c \in \mathbb{R}$. *Then, the following properties are satisfied:*

(i) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} \, ,$

(ii) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \, ,$

(iii) $c\,(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v}) \, ,$

(iv) $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$ *, and*

(v) $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$, *where* θ *is the angle between* $\vec{u}$ *and* $\vec{v}$.

Definition 1.15 *Two vectors are said to be* <u>*orthogonal*</u> *(or,* <u>*perpendicular*</u>*) if the angle between them is* $\pm\pi/2$.

Proposition 1.16 *Two vectors* $\vec{u}$ *and* $\vec{v}$ *are orthogonal if and only if* $\vec{u} \cdot \vec{v} = 0$.

1.2 The Cross Product

Definition 1.17 *The* <u>*cross product*</u> *(or, also* <u>*outer*</u> *or* <u>*vector*</u> *product) between two vectors* $\vec{u} = \langle u_1, u_2, u_3 \rangle$ *and* $\vec{v} = \langle v_1, v_2, v_3 \rangle$ *is the vector given by*

$$\vec{u} \times \vec{v} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle \, .$$

Remark 1.18 *In order to remember about definition is useful to understand the cross product as the following formal determinant:*

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \, .$$

Proposition 1.19 (Properties of the Cross Product) *Let* $\vec{u}$ *and* $\vec{v}$ *be two vectors. Then, the following properties are satisfied:*

(i) The cross product $\vec{u} \times \vec{v}$ *is orthogonal to both* $\vec{u}$ *and* $\vec{v}$.

(ii) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$ *, and*

(iii) $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$*, where* $\theta \in [0, \pi)$ *is the angle between* $\vec{u}$ *and* $\vec{v}$*.*

Theorem 1.20 *The norm of the cross product* $\vec{u} \times \vec{v}$ *is the area of the parallelogram with sides* $\vec{u}$ *and* $\vec{v}$*. Similarly, the area of a triangle with sides* $\vec{u}$ *and* $\vec{v}$ *is, precisely,* $\|\vec{u} \times \vec{v}\|/2$*.*

Definition 1.21 *The triple product of* $\vec{u}$*,* $\vec{v}$ *and* $\vec{w}$ *is the number* $\vec{u} \cdot (\vec{v} \times \vec{w})$*.*

Theorem 1.22 *The absolute value of the triple product (i.e.,* $|\vec{u} \cdot (\vec{v} \times \vec{w})|$*) represents the volume of the parallelepiped with sides* $\vec{u}$*,* $\vec{v}$ *and* $\vec{w}$*.*

1.3 Lines in $\mathbb{R}^3$

Definition 1.23 *A line is an affine subspace of dimension one.*

Proposition 1.24 (The First Postulate of Euclide) *For every two points in space there is a line passing through them, and such a line is unique.*

Proposition 1.25 *Let* $P = (p_1, p_2, p_3)$ *and* $Q = (q_1, q_2, q_3)$ *be two points in space and denote by* $\vec{v} = \vec{PQ}$ *the director vector. Then, the unique line passing through* P *and* Q *can be represented as:*

1. *Vector Equation.*
$$\vec{r}(t) = P + t\vec{v} = \langle p_1 + tv_1, p_2 + tv_2, p_3 + tv_3 \rangle \,,$$
where t *is the parameter of the line.*

2. *Parametric Equations.*
$$\begin{cases} x = p_1 + tv_1 \,, \\ y = p_2 + tv_2 \,, \\ z = p_3 + tv_3 \,. \end{cases}$$

3. *Symmetric Equations. Assuming* $v_1, v_2, v_3 \neq 0$*,*
$$\frac{x - p_1}{v_1} = \frac{y - p_2}{v_2} = \frac{z - p_3}{v_3} \,.$$
(If any v_i*,* $i = 1, 2, 3$ *is zero, the line is contained in a plane.)*

4. *Implicit Equations.*
$$\begin{cases} ax + by + cz = d \,, \\ \widetilde{a}x + \widetilde{b}y + \widetilde{c}z = \widetilde{d} \,. \end{cases}$$

Example 1.26 *Write the equations of the line passing through the points* $P = (1, -1, 0)$ *and* $Q = (2, 1, -1,)$*.*

Proposition 1.27 *Two lines in the space can be:*

(i) *The same.*

(ii) *Parallel.*

(iii) *Skew.*

(iv) *Intersecting (the intersection is a point).*

Let $\vec{v}_1$ and $\vec{v}_2$ be the director vectors, respectively. If $\vec{v}_1$ is a multiple of $\vec{v}_2$ (i.e., there exists a real number $c \neq 0$ such that $\vec{v}_1 = c\vec{v}_2$, they are proportional), then the two lines are either the same or parallel. In the other case, the lines are:

- *Skew, if they do not share any points.*
- *Intersecting, if there exists a common point.*

Example 1.28 *Determine the relative position of the following pair of lines:*

1.
$$\begin{cases} \frac{x-1}{2} = y+1 = \frac{z-2}{4}\,, \\ \frac{x+2}{4} = \frac{-y}{3} = z+1\,. \end{cases}$$

(Answer: Skew.)

2.
$$\begin{cases} \frac{x-1}{2} = y+1 = \frac{z-2}{4}\,, \\ \frac{x+2}{4} = \frac{-y}{3} = \frac{-2z+1}{2}\,. \end{cases}$$

(Answer: Skew.)

1.4 Planes in $\mathbb{R}^3$

Definition 1.29 *A plane is an affine subspace of dimension two.*

Proposition 1.30 *For every three non-aligned points in space there is a unique plane passing through them. Equivalently, it is enough to know a point and two non proportional vectors, or a point and a normal vector.*

Proposition 1.31 *Let $P = (p_1, p_2, p_3)$, $Q = (q_1, q_2, q_3)$ and $R = (r_1, r_2, r_3)$ be three non-aligned points in space and denote by $\vec{v} = \vec{PQ}$ and by $\vec{u} = \vec{PR}$. Then, the unique plane passing through P, Q and R can be represented as:*

1. *Vector Equation.*

$$\pi(s,t) = P + t\vec{v} + s\vec{u} = \langle p_1 + tv_1 + su_1, p_2 + tv_2 + su_2, p_3 + tv_3 + su_3 \rangle \,,$$

   where t and s are the parameters of the plane.

2. *Parametric Equations.*

$$\begin{cases} x = p_1 + tv_1 + su_1 \,, \\ y = p_2 + tv_2 + su_2 \,, \\ z = p_3 + tv_3 + su_3 \,. \end{cases}$$

3. *Implicit Equations. Let $\vec{N} = \langle a, b, c \rangle$ be any normal vector to the plane (a normal vector can be computed as the cross product of $\vec{v}$ and $\vec{u}$), then the implicit equation of the plane is:*

$$ax + by + cz = d \,.$$

Example 1.32 *Represent the plane passing through the points $P = (1,1,1)$, $Q = (2,4,3)$ and $R = (-1,-2,-1)$.*

Proposition 1.33 *Two planes in the space can be:*

(i) The same.

(ii) Parallel.

(iii) Intersecting (the intersection is a line).

Let $\vec{N}_1$ and $\vec{N}_2$ be the normal vectors, respectively. If they are proportional, then the two planes are either the same or parallel. In the other case, the planes intersect in a line.

Proposition 1.34 *Every line is the intersection of two planes.*

Example 1.35 *Determine the relative position of the following planes:*

$$\begin{cases} 2x + y - 4z = 3 \,, \\ x - y + z = 2 \,. \end{cases}$$

(Answer: Intersecting.)

1.5 Distance

Definition 1.36 *The* <u>*distance*</u> *between two affine subspaces (points, lines, planes) is the minimum of the distances between all the possible pair of points, i.e.,*

$$d(A, B) = \min\{d(P, Q) \mid P \in A, Q \in B\}.$$

Example 1.37 *Compute the distance between the following subspaces:*

1. *Distance from the point* $P = (0, 3, 6)$ *to the line* $\vec{r}(t) = \langle 1 - t, 1 + 2t, 5 + 3t\rangle$*.*
   (Answer: $d(P, \mathcal{L}) = d(P, Q) = \sqrt{70}/7$*, where* $Q = (3/7, 15/7, 47/7)$*.)*

2. *Distance between the point* $P = (1, 1, 1)$ *and the plane of equation* $x - y + z = 2$*.*
   (Answer: $d(P, \pi) = d(P, Q) = \sqrt{3}/3$*, where* $Q = (4/3, 2/3, 4/3)$*.)*

3. *Distance between the lines*

$$\mathcal{L}_1 : \begin{cases} x = 2t + 1 \\ y = t + 1 \\ z = -t + 1 \end{cases}, \qquad \text{and} \qquad \mathcal{L}_2 : \begin{cases} x - 2z + 1 = 0 \\ x - y + z = 0 \end{cases}.$$

   (Answer: $d(\mathcal{L}_1, \mathcal{L}_2) = d(P, Q) = \sqrt{3}/3$*, where* $P = (1/2, 5/4, 3/4)$ *and* $Q = (5/6, 11/12, 13/12)$*.)*

1.6 Exercises

1. Find the equation of the plane passing through $P = (2, 5, 3)$ and perpendicular to the vector $\vec{N} =< 1, -4, -3 >$. (Answer: $x - 4y - 3z = -27$.)

2. Find the equation of the plane through the point $P = (2, 5, 3)$ and parallel to the plane $x - 4y - 3z = -8$. (Answer: $x - 4y - 3z = -27$.)

3. Find a nonzero vector parallel to the line of intersection of the two planes $x - 4y - 3z = -4$ and $-3x + 3y - z = 3$. (Answer: $< -13, -10, 9 >$.)

4. Find the implicit equation for the plane through $P = (-3, 3, -1)$ and normal to the vector $< 1, -4, -3 >$. (Answer: $x - 4y - 3z = -12$.)

5. Find the equation of the plane passing through the points $P = (-4, 3, 0)$, $Q = (-3, -1, -3)$ and $R = (-7, 6, -1)$. (Answer: $13x + 10y - 9z = -22$.)

2 Vector-Valued Functions (Chapter 10)

Definition 2.1 *Let $f(t)$, $g(t)$ and $h(t)$ be usual functions in the variable t. A vector-valued function $\vec{R}(t)$ is a function of the type*

$$\vec{R}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}\,,$$

and its image for a fixed t is a vector.

Example 2.2 *The vector parametric equation of a line,*

$$\vec{r}(t) = \langle p_1 + tv_1, p_2 + tv_2, p_3 + tv_3 \rangle\,,$$

is a vector valued-function.

Example 2.3 *Compute the inner product $\vec{R} \cdot \vec{F}$ and the cross product $\vec{R} \times \vec{F}$ of the following vector valued functions:*

$$\vec{R}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}\,, \qquad \vec{F}(t) = \cos t\,\vec{i} - t \sin t\,\vec{k}\,.$$

Remark 2.4 *Vector-valued functions can be understood as the parameterizations of a curve. Consequently, the vector-valued function $\vec{R}(t)$ is usually called the position vector.*

Definition 2.5 *The velocity vector of a curve is the first derivative of the position vector $\vec{R}(t)$ with respect to the parameter t, i.e., $\vec{R}\,'(t)$. The speed is the norm of the velocity vector, i.e., $\|\vec{R}\,'(t)\|$. And, the acceleration is the second derivative of the position vector (or, equivalently, the first derivative of the velocity vector) with respect to the parameter t, i.e., $\vec{R}\,''(t)$.*

Example 2.6 *Consider the position vector of a line, i.e.,*

$$\vec{r}(t) = (p_1 + tv_1)\,\vec{i} + (p_2 + tv_2)\,\vec{j} + (p_3 + tv_3)\,\vec{k}\,.$$

The velocity vector is

$$\vec{r}\,'(t) = \langle v_1, v_2, v_3 \rangle\,,$$

that is, precisely, the director vector. While, the acceleration is identically zero, i.e., $\vec{r}\,''(t) = \langle 0, 0, 0 \rangle$.

Example 2.7 *Compute the velocity and the acceleration of the curve parameterized by*

$$\vec{R}(t) = (1 - 2t)\,\vec{i} - t^2\,\vec{j} + e^t\,\vec{k}\,,$$

at $t_o = 0$.
(Answer: $\vec{V}(0) = \langle -2, 0, 1 \rangle$ and $\vec{A}(0) = \langle 0, -2, 0 \rangle$.)

Example 2.8 *Given the acceleration*

$$\vec{A}(t) = \cos t\,\vec{\imath} - t\sin t\,\vec{k}$$

the initial velocity $\vec{V}(0) = \langle 1, -2, 1\rangle$ *and initial position* $\vec{R}(0) = \langle 0, 2, 3\rangle$*, find the position vector* $\vec{R}(t)$*.*
(Answer: $\vec{R}(t) = (-\cos t + t + 1)\,\vec{\imath} + (t^2/2 - 2t + 2)\,\vec{\jmath} + (2\cos t + t\sin t + t + 1)\,\vec{k}$*.)*

Definition 2.9 *Given a curve parameterized by a vector-valued function* $\vec{R}(t)$*, the* <u>*unit tangent*</u> *vector is given by*

$$T(t) = \frac{\vec{R}\,'(t)}{\|\vec{R}\,'(t)\|}\,.$$

The <u>*unit normal*</u> *vector is given by*

$$N(t) = \frac{T\,'(t)}{\|T\,'(t)\|}\,,$$

and the <u>*unit binormal*</u> *is* $B(t) = T(t) \times N(t)$*. We call* $\{T(t), N(t), B(t)\}$ *the* <u>*Frenet frame*</u>*.*

Definition 2.10 *The* <u>*curvature*</u> $\kappa(t)$ *of a curve is given by the following formula*

$$\kappa(t) = \frac{\|\vec{R}\,'(t) \times \vec{R}\,''(t)\|}{\|\vec{R}\,'(t)\|^3}\,.$$

Example 2.11 *Compute the curvature of the curve parameterized by*

$$\vec{R}(t) = \cos t\,\vec{\imath} + \sin t\,\vec{\jmath} + t\,\vec{k}\,.$$

2.1 Exercises

1. Find the position vector $\vec{R}(t)$ given the velocity $\vec{V}(t) = e^t\vec{\imath} + 8\cos(4t)\vec{\jmath} + 16t^3\vec{k}$ and the initial position $\vec{R}(0) =< 2, -2, 3 >$.
   (Answer: $\vec{R}(t) = (e^t + 1)\,\vec{\imath} + 2\,(\sin(4t) - 1)\,\vec{\jmath} + (4t^4 + 3)\,\vec{k}$.)
2. Find the curvature of the planar curve $y = 2x + 1/x$ at $x = 1$. (Answer: $\kappa = 1/\sqrt{2}$.)
3. Find the curvature of the planar curve $y = 3e^x$ at $x = 0$. (Answer: $\kappa = 3/(\sqrt{10})^3$.)
4. Find the curvature of the curve parameterized by $\vec{R}(t) = 2\sin(2t)\vec{\imath} + 2\cos(2t)\vec{\jmath} + \vec{k}$.
5. Find the curvature of the curve parameterized by $\vec{R}(t) = t^2\vec{\imath} - te^t\vec{\jmath} + t^2\cos(t)\vec{k}$ at $t = 0$.

3 Partial Differentiation (Chapter 11)

3.1 Functions of Several Variables

Definition 3.1 *A function of several variables $f(x_1, ..., x_n)$ is a map from a subset $\mathcal{D}$ of $\mathbb{R}^n$ to the real numbers, i.e., it is a scalar-valued function. The subset $\mathcal{D}$ is the domain of the function. The range of f is the set of real numbers that can be written as $f(x_1, ..., x_n)$.*

Remark 3.2 *In particular, we will focus on functions of two variables $f(x, y)$.*

Example 3.3 *Consider the function $f(x, y) = 1/\sqrt{x^2 - y^2}$. Compute the domain and range of f.*

Remark 3.4 *The graph $z = f(x, y)$ of a function of two variables is a surface in space. Locally, any surface can be described as a graph of a function.*

Definition 3.5 *A level curve for the constant value $c \in \mathbb{R}$ is the set of points in the plane satisfying $f(x, y) = c$.*

Definition 3.6 *A quadric is a surface which is described by an equation of the form*

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0\,.$$

Proposition 3.7 *Quadrics can be classified in the following groups:*

(i) Cylinders.

(ii) Ellipsoids:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,.$$

In particular, if $a = b = c = R$ we have a sphere of radius R.

(iii) Cones:

$$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}\,.$$

(iv) Hyperboloids:

- *Hyperboloid of one sheet:*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1\,.$$

- *Hyperboloid of two sheets:*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1\,.$$

(v) Paraboloids:

- *Elliptic paraboloid:*
$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}\,.$$
- *Hyperbolic paraboloid:*
$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}\,.$$

3.2 Limits of Functions in Two Variables

Definition 3.8 *Let $f : \mathcal{D} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function of two variables and $(a, b) \in \mathcal{D}$. We say that the limit of $f(x, y)$ when (x, y) tends to (a, b) is L,*

$$\lim_{(x,y)\to(a,b)} f(x, y) = L\,,$$

if when (x, y) is "close" to (a, b), the value $f(x, y)$ is "close" to L.

Remark 3.9 *For functions of two variables there are more options of being "close" than for functions of one variable, i.e., we can approach the point (a, b) in many ways.*

Definition 3.10 *The limit $\lim_{(x,y)\to(a,b)} f(x, y)$ exists if the function $f(x, y)$ approaches the same value L along every possible way of approaching (a, b).*

Proposition 3.11 (Iterated Limits) *If the iterated limits*

$$\lim_{x\to a}\big(\lim_{y\to b} f(x, y)\big), \qquad \text{and} \qquad \lim_{y\to b}\big(\lim_{x\to a} f(x, y)\big),$$

are not equal (or one of them is $\pm\infty$), then the limit

$$\lim_{(x,y)\to(a,b)} f(x, y)\,,$$

does not exist. (However, if they coincide we need to keep working.)

Proposition 3.12 (Limits Along Lines) *Consider the general equation of all lines passing through the point (a, b), $y = m\,(x - a) + b$. If the limit*

$$\lim_{x\to a} f(x, m[x - a] + b)\,,$$

depends on the constant m, then the limit

$$\lim_{(x,y)\to(a,b)} f(x, y)\,,$$

does not exist. (However, if they coincide we need to keep working.)

Proposition 3.13 (Polar Coordinates) *Consider the parameterization of the circles of radii r centered at (a,b) given by the polar coordinates,*

$$\begin{cases} x = a + r\cos\theta\,, \\ y = b + r\sin\theta\,. \end{cases}$$

Then,

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{r\to 0} f(a + r\cos\theta, b + r\sin\theta)\,.$$

In particular, if above limit depends on θ, then the limit

$$\lim_{(x,y)\to(a,b)} f(x,y)\,,$$

does not exist. (Otherwise, polar coordinates give us the answer.)

Example 3.14 *Compute the following limits:*

1. $$\lim_{(x,y)\to(0,0)} \frac{2x^2}{x^2+y^2}\,.$$
2. $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}\,.$$
3. $$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2}\,.$$
4. $$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}\,.$$
5. $$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{2\sqrt{x^2+y^2+9}-6}\,.$$

3.3 Continuity of Functions in Two Variables

Definition 3.15 *We say that a function of two variables $f(x,y)$ is* <u>*continuous*</u> *at a point (a,b) in its domain if:*

(i) $f(a,b)$ *exists,*

(ii) $\lim_{(x,y)\to(a,b)} f(x,y)$ *exists, and*

(iii) $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

We say that $f(x,y)$ is <u>continuous</u> in a domain $\mathcal{D}$ if it is continuous at all $(a,b) \in \mathcal{D}$.

Remark 3.16 *All the properties about continuous functions in one variable extend to two variables:*

1. *The sum of continuous functions is continuous.*
2. *The product of continuous functions is continuous.*
3. *The composition of continuous functions is continuous.*

Example 3.17 *Study the continuity of the following functions:*

1. $$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}\,, & (x,y) \neq (0,0)\,, \\ 0\,, & (x,y) = (0,0)\,. \end{cases}$$

2. $$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2y^2+(x-y)^2}\,, & (x,y) \neq (0,0)\,, \\ 0\,, & (x,y) = (0,0)\,. \end{cases}$$

3. $$f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2}\,, & (x,y) \neq (0,0)\,, \\ A\,, & (x,y) = (0,0)\,. \end{cases}$$

4. $$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}\,, & (x,y) \neq (0,0)\,, \\ 0\,, & (x,y) = (0,0)\,. \end{cases}$$

3.4 Partial and Directional Derivatives

Definition 3.18 *Let $f(x,y)$ be a function in two variables. The <u>partial derivative</u> of f with respect to x is given by:*

$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}\,.$$

Similarly, the partial derivative of f with respect to y is:

$$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}\,.$$

Remark 3.19 *From the definition, it is clear that for f_x we can differentiate assuming y is constant. And, similarly for f_y.*

Example 3.20 *Compute the partial derivatives of the following functions:*

(i) $f(x,y) = 4x^2 + 2xy - y^2 + 3x - 2y + 5$

(ii) $f(x,y) = \cos(x^3 - 3x^2y^2 + 2y^4)$

Definition 3.21 *The* <u>*gradient*</u> *of f is the vector field given by*

$$\nabla f(a,b) = f_x(a,b)\vec{i} + f_y(a,b)\vec{j}\,.$$

Remark 3.22 *Geometrically, f_x represents the slope of the line tangent to the graph $z = f(x,y)$ parallel to the x-axis. Similarly, for f_y.*

Remark 3.23 *The gradient points in the direction of maximum rate of change.*

Definition 3.24 *Let $S \subset \mathbb{R}^3$ be a surface and let $P = (a, b, f(a,b)) \in S$ be a point in the surface. The* <u>*tangent plane*</u> *at P is given by the implicit equation*

$$z = f(a,b) + f_x(a,b)\,(x-a) + f_y(a,b)\,(y-b)\,.$$

Example 3.25 *Find the tangent plane to the surface that is the graph of $f(x,y) = x^3 - x^2y + y^2 - 2x + 3y$ at $(a,b) = (-1,3)$.*

Definition 3.26 *Let $\vec{v} =< v_1, v_2 >$ be a unit vector in the plane $\mathbb{R}^2$. The* <u>*directional derivative*</u> *of f with respect to $\vec{v}$ is:*

$$D_{\vec{v}} f = \lim_{h \to 0} \frac{f(x + hv_1, y + hv_2) - f(x,y)}{h}\,.$$

In general, for an arbitrary vector $\vec{u}$ (unit or not) we compute:

$$D_{\vec{u}} f = \nabla f \cdot \frac{\vec{u}}{\|\vec{u}\|}\,.$$

Example 3.27 *Compute the directional derivatives of the following functions $f(x,y)$:*

(i) $f(x,y) = \ln(x^2 + y^2)$ *at* $P = (1,-3)$ *in the direction of* $\vec{u} =< 2, -3 >$.

(ii) $f(x,y) = x^2 - y^2$ *at* $P = (1,0)$ *in the direction of* $\vec{u} =< \sqrt{3}/2, 1/2 >$.

3.5 Differentiability

Definition 3.28 *A function $f(x,y)$ is* <u>*differentiable*</u> *at a point (a,b) if*

$$\lim_{(x,y) \to (a,b)} \frac{f(x,y) - f(a,b) - f_x(a,b)\,(x-a) - f_y(a,b)\,(y-b)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0\,.$$

Remark 3.29 *Roughly speaking, if the function $f(x,y)$ at (a,b) can be approximated by a linear function, the tangent plane.*

Example 3.30 *Is the function $f(x,y) = 3x - 4y^2$ differentiable at $(-1,2)$?*

Proposition 3.31 *If $f(x,y)$ is differentiable at (a,b), then $f(x,y)$ is continuous at (a,b).*

Theorem 3.32 *Let $f(x,y)$ be a function in two variables and $(a,b) \in \mathcal{D}$ a point in the domain of f. If the partial derivatives of f, f_x and f_y exist and are continuous at (a,b), then f is differentiable at (a,b).*

Remark 3.33 *A function can have partial derivatives, without being differentiable. For instance,*

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}\,, & (x,y) \neq (0,0)\,, \\ 0\,, & (x,y) = (0,0)\,. \end{cases}$$

3.6 Higher Order Partial Derivatives

Remark 3.34 *Given a function $f(x,y)$ its partial derivatives f_x and f_y are also functions in two variables, so we can compute their partial derivatives:*

$$\begin{aligned} f_{xx} &= \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \\ f_{xy} &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \\ f_{yx} &= \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \\ f_{yy} &= \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right), \end{aligned}$$

and we can keep going, i.e., f_{xxx},...

Theorem 3.35 (Clairut-Schwarz) *Let $f(x,y)$ be a function of two variables and assume that f has continuous second order partial derivatives at a point (a,b), then*

$$\frac{\partial^2 f}{\partial x \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b)\,.$$

Remark 3.36 *Continuity is essential in above result. If not, the following function serves as a counterexample:*

$$f(x,y) = \begin{cases} \frac{x^3y - y^3x}{x^2+y^2}\,, & (x,y) \neq (0,0)\,, \\ 0\,, & (x,y) = (0,0)\,. \end{cases}$$

Example 3.37 *Compute the second order partial derivatives of the function $f(x,y) = \sin(3x - 2y) + e^{x+4y}$.*

3.7 The Chain Rule

Proposition 3.38 *Let $x(u,v)$ and $y(u,v)$ be differentiable functions in the variables u and v, and assume that $f(x,y)$ is a differentiable function in x and y. Then,*

$$f\left(x(u,v), y(u,v)\right)$$

is a differentiable function in u and v and

$$\begin{aligned}\frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,, \\ \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.\end{aligned}$$

Example 3.39 *Let $f(x,y) = 3x^2 - 2xy$, $x(u,v) = e^2 \sin v$ and $y(u,v) = u + 2v + e^{uv}$. Compute f_u and f_v.*

Remark 3.40 *The chain rule can be used to implicitly differentiate a function.*

Example 3.41 *Let $z \equiv z(x,y)$ given by the implicit equation $x^2z + yz^3 = x$. Compute the partial derivatives z_x and z_y.*

Example 3.42 *Find the tangent plane to the surface $4x^2 - 2y^2 + z^2 = 12$ at $(2,2,2)$.*

3.8 Extrema of Functions of Two Variables

Definition 3.43 *Let $f(x,y)$ be a differentiable function of two variables. We say that a point $(a,b) \in \mathcal{D}$ in the domain of f is a <u>critical point</u> if either:*

(i) $\nabla f(a,b) =< 0, 0 >= 0$, *or,*

(ii) one of the partial derivatives, f_x or f_y, does not exist.

Example 3.44 *Find the critical points of the function $f(x,y) = \sqrt{1 - x^2 - y^2}$.*

Definition 3.45 *Let $f(x,y)$ be a differentiable function of two variables. We say that a point $(a,b) \in \mathcal{D}$ is:*

(i) A <u>local maximum</u> for f, if $f(x,y) \leq f(a,b)$ for all (x,y) "close" to (a,b).

(ii) A <u>global maximum</u> for f, if $f(x,y) \leq f(a,b)$ for all (x,y).

(iii) A <u>local minimum</u> for f, if $f(x,y) \geq f(a,b)$ for all (x,y) "close" to (a,b).

(iv) A <u>global minimum</u> for f, if $f(x,y) \geq f(a,b)$ for all (x,y).

Remark 3.46 *Clearly global extrema are also local/relative.*

Theorem 3.47 *Let $f(x,y)$ be a differentiable function of two variables and assume that $f_x(a.b)$ and $f_y(a,b)$ exist. If $(a,b) \in \mathcal{D}$ is a local maximum (or, minimum), then*

$$\nabla f(a,b) =< 0, 0 >= 0\,.$$

Remark 3.48 *The converse is not true. For instance, the function $f(x,y) = x^2 - y^2$ has a critical point at $(0,0)$ which is not a local maximum nor a local minimum.*

Definition 3.49 *A <u>saddle point</u> is a critical point which is not a local maximum nor a local minimum.*

Remark 3.50 *To understand the local nature of critical points we need to check, at least, the second order partial derivatives.*

Definition 3.51 *The <u>Hessian matrix</u> of a function $f(x,y)$ at a point $(a,b) \in \mathcal{D}$ is*

$$\mathcal{H}f(a,b) = \begin{pmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{pmatrix}\,.$$

We will denote by $D = \det \mathcal{H}f(a,b)$ to the determinant of the Hessian matrix of f at (a,b). (That is, $D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)f_{yx}(a,b)$.)

Theorem 3.52 (Second Partial Derivative Test) *Let $f(x,y)$ be a differentiable function and assume that (a,b) is a critical point where $\nabla f(a,b) =< 0, 0 >$. Then:*

(i) If $D > 0$ and $f_{xx}(a,b) > 0$, the point (a,b) is a local minimum.

(ii) If $D > 0$ and $f_{xx}(a,b) < 0$, the point (a,b) is a local maximum.

(iii) If $D < 0$, the point (a,b) is a saddle point.

(iv) If $D = 0$, the test is "inconclusive".

Example 3.53 *Compute the critical points of the following functions and classify them:*

1. $f(x,y) = x^3 + 2xy - 6x - 4y^2$
2. $f(x,y) = 2x^2 + 2xy + y^2 - 2x + 5$
3. $f(x,y) = x^2y^4$
4. $f(x,y) = 8x^3 - 24xy + y^3$

Theorem 3.54 (Global Extrema) *A continuous function defined on a compact (bounded and closed) domain attains both its global minimum and global maximum.*

Definition 3.55 *A set is* <u>*closed*</u> *if the boundary is included.*

Remark 3.56 *How to find the global extrema?*

1. *Compute the critical points in the interior of the domain. (The interior is the set without the boundary.)*
2. *Determine the maximum and minimum value of the function at the boundary. (One variable problem.)*
3. *Compare the values of the function at the following points: the points of 1), the points of 2) and the points in the boundary of the boundary.*

Example 3.57 *Compute the extrema of the following functions:*

1. $f(x,y) = e^{x^2-y^2}$ *defined on* $\{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 \leq 1\}$.
2. $f(x,y) = \frac{9x}{x^2+y^2+1}$ *for* $x \in [-1,1]$ *and* $y \in [-1,1]$.
3. $f(x,y) = 4x^2 - 2xy + 6y^2 - 8x + 2y + 3$, *for* $(x,y) \in [0,2] \times [-1,3]$.
4. $f(x,y) = x^2 + y^2 - 2y + 1$ *defined on* $\{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 \leq 4\}$.

Remark 3.58 *Step 2 in previous remark is usually hard if the domain has a weird shape (other than a circle, an ellipse or a square). To solve this problem we may need the Lagrange multipliers principle.*

3.9 Lagrange Multipliers

Remark 3.59 *Lagrange multipliers serve to compute maxima and minima of a differentiable function* $f(x,y)$ *subject to a constraint* $g(x,y) = 0$.

Theorem 3.60 (Lagrange Multipliers Principle) *Let* $f(x,y)$ *and* $g(x,y)$ *be functions in two variables with continuous partial derivatives and assume that* $f(x,y)$ *restricted to the curve* $g(x,y) = 0$ *has a local extrema* (a,b) *such that* $\nabla g(a,b) \neq < 0,0 >$. *Then, there exists a number* $\lambda \in \mathbb{R}$ *such that*

$$\nabla f(a,b) = \lambda \nabla g(a,b)\,.$$

Definition 3.61 *The number* $\lambda \in \mathbb{R}$ *is referred as to the* <u>*Lagrange multiplier*</u>.

Remark 3.62 *If we want to find extrema of a function $f(x, y)$ subject to a constraint $g(x, y) = 0$, from Lagrange multipliers principle, we need to solve*

$$\begin{cases} \nabla f(x, y) = \lambda \nabla g(x, y)\,, \\ g(x, y) = 0\,, \end{cases}$$

to obtain the candidates. Observe that this is a system of three equations (possibly, nonlinear) with three unknowns, i.e., x, y and λ.

Example 3.63 *Compute maxima and minima of the following functions with their respective constraints:*

1. $f(x, y) = x + y$ *subject to* $g(x, y) = x^2 + y^2 - 1 = 0$.
2. $f(x, y) = 1 - x^2 - y^2$ *subject to* $g(x, y) = x + y - 1 = 0$.
3. $f(x, y) = x^2 + y^2$ *subject to* $g(x, y) = x^2 + xy + y^2 - 1 = 0$.
4. $f(x, y) = e^{x^2 - y^2}$ *subject to* $g(x, y) = x^2 + y^2 - 1 = 0$.
5. $f(x, y) = x^2 + y^2 - 2y + 1$ *subject to* $g(x, y) = x^2 + y^2 - 4 = 0$.

Review Problems

1. Compute the distance from the point $P = (3, -1, 4)$ to the line

$$\mathcal{L} : \begin{cases} x = t\,, \\ y = 2 + t\,, \\ z = 1 - t\,. \end{cases}$$

   (Use the family of planes orthogonal to $\mathcal{L}$ to find the closest point in $\mathcal{L}$ to P.)

2. Compute the distance from the point $P = (1, 1, 2)$ to the line $\mathcal{L}(t) = \langle -2+3t, 1-t, 3+2t\rangle$. (Use the family of planes orthogonal to $\mathcal{L}$ to find the closest point in $\mathcal{L}$ to P.)

3. Compute the distance from the point $P = (-2, 1, 0)$ to the plane $\pi : x - y + 2z = 7$. (Use the line perpendicular to π to find the closest point in π to P.)

4. Determine the relative position between the lines

$$\mathcal{L}_1 : \begin{cases} x = 2 - t\,, \\ y = 1 + t\,, \\ z = -2 - t\,, \end{cases} \qquad \mathcal{L}_2 : \begin{cases} x = 1 + s\,, \\ y = s\,, \\ z = 5 + s\,. \end{cases}$$

   (If they are skew, compute the distance between them finding the closest points P and Q.)

5. Determine the relative position between the lines

$$\mathcal{L}_1 : \begin{cases} x = 2t + 3\,, \\ y = t + 2\,, \\ z = -t\,, \end{cases} \qquad \mathcal{L}_2 : \begin{cases} x - y + z = 0\,, \\ 2x - y - z + 1 = 0\,. \end{cases}$$

   (If they are skew, compute the distance between them finding the closest points P and Q.)

6. Compute the position vector given the acceleration $\vec{A}(t) = -4\cos(2t)\vec{\imath} + 4\sin(2t)\vec{\jmath} - 2t\vec{k}$ and initial velocity and position $\vec{V}(0) = \langle 1, 0, 1\rangle$ and $\vec{R}(0) = \langle 1, 1, 1\rangle$, respectively.

7. Compute the position vector given the acceleration $\vec{A}(t) = \cos(t)\vec{\imath} + \sin(t)\vec{\jmath} + 3t\vec{k}$ and initial velocity and position $\vec{V}(0) = \langle 1, 0, 1\rangle$ and $\vec{R}(0) = \langle 1, 1, 1\rangle$, respectively.

8. Compute the acceleration and position vector given the velocity $\vec{V}(t) = \sqrt{t}\,\vec{\imath} - e^t\vec{\jmath} + \sin t\vec{k}$ and initial position $\vec{R}(0) = \langle 1, -1, 2\rangle$.

9. Compute the acceleration and position vector given the velocity $\vec{V}(t) = t^2\vec{\imath} - \sin(2t)\vec{\jmath} + 2te^{t^2}\vec{k}$ and initial position $\vec{R}(0) = \langle 1, -1/2, 0\rangle$.

10. Compute the acceleration and position vector given the velocity $\vec{V}(t) = e^{3t}\vec{\imath} - 2\sin(2t)\vec{\jmath} + \sqrt{t}\vec{k}$ and initial position $\vec{R}(0) = \langle 1/3, -1, 1\rangle$.

11. Study the continuity of the following function:

$$f(x,y) = \begin{cases} \frac{4xy^2}{x^2+3y^4}\,, & (x,y) \neq (0,0)\,, \\ 0\,, & (x,y) = (0,0)\,. \end{cases}$$

12. Study the continuity of the following function:

$$f(x,y) = \begin{cases} \frac{3xy}{x^2+y^2}\,, & (x,y) \neq (0,0)\,, \\ 0\,, & (x,y) = (0,0)\,. \end{cases}$$

13. Study the continuity of the following function:

$$f(x,y) = \begin{cases} \frac{2(x^2+y^2)}{\sqrt{x^2+y^2+4}-2}\,, & (x,y) \neq (0,0)\,, \\ 8\,, & (x,y) = (0,0)\,. \end{cases}$$

14. Study the continuity of the following function:

$$f(x,y) = \begin{cases} \frac{x-y}{x^2+4y^4}\,, & (x,y) \neq (0,0)\,, \\ A\,, & (x,y) = (0,0)\,. \end{cases}$$

15. Study the continuity of the following function:

$$f(x,y) = \begin{cases} \frac{x^2y}{x^3+y^2}\,, & (x,y) \neq (0,0)\,, \\ 0\,, & (x,y) = (0,0)\,. \end{cases}$$

16. Let $f(x,y) = x^2 + 2xy + 3y^2 - 4x - 3y$. Obtain critical points and classify them as local maximum, local minimum or saddle points.

17. Let $f(x,y) = x^2 - 3xy + y^2$. Obtain critical points and classify them as local maximum, local minimum or saddle points.

18. Let $f(x,y) = x^2 + 3xy + 5y^3 - 7x - 11y$. Obtain critical points and classify them as local maximum, local minimum or saddle points.

19. Let $f(x,y) = x^3 - 3x + y^3 - 3y^2$. Obtain critical points and classify them as local maximum, local minimum or saddle points.

20. Let $f(x,y) = 2x^2 + y^2 + 6y$ be defined on $\mathbb{R}^2$. Obtain global extrema, if possible.

21. Let $f(x,y) = 2x^2 - y^2 + 6y$ be defined on $\{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 16\}$. Obtain extrema.

22. Let $f(x,y) = x^2 + 3y^2$ be defined on $\mathbb{R}^2$. Obtain global extrema, if possible.

23. Let $f(x, y) = x^2 + 3y^2$ be defined on $\{(x, y) \in \mathbb{R}^2 \,|\, (x-1)^2 + y^2 \leq 4\}$. Obtain extrema.

24. Let $f(x, y) = 4x^2 + y^2 - 4x$ be defined on $\mathbb{R}^2$. Obtain global extrema, if possible.

25. Let $f(x, y) = 4x^2 + y^2 - 4x$ be defined on $\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}$. Obtain extrema.

26. Let $f(x, y) = x^2 + y^2 + 2x - 1$ be defined on $\mathbb{R}^2$. Obtain global extrema, if possible.

27. Let $f(x, y) = x^2 + y^2 + 2x - 1$ be defined on $\{(x, y) \in \mathbb{R}^2 \,|\, 2x^2 + y^2 \leq 8\}$. Obtain extrema.

28. Let $f(x, y) = -x^2 - y^2 + 2x - 1$ be defined on $\mathbb{R}^2$. Obtain global extrema, if possible.

29. Let $f(x, y) = -x^2 - y^2 + 2x - 1$ be defined on $\{(x, y) \in \mathbb{R}^2 \,|\, 2x^2 + y^2 \leq 8\}$. Obtain extrema.

4 Multiple Integration (Chapter 12)

4.1 Double Integrals Over Rectangular Regions

Definition 4.1 *Let $f(x,y)$ be defined on a closed, bounded rectangular region R in the xy-plane. The double integral of f over R is defined by*

$$\int\int_R f(x,y)\,dA = \lim_{\|P\|\to 0} \sum_{k=1}^{N} f(x_k^*, y_k^*)\Delta A_k\,,$$

where A_k is the area of the k-th representative cell and $\|P\|$ is the norm of the partition, i.e., the length of the longest diagonal of any rectangle in the partition.

Definition 4.2 *The sum $\sum_{k=1}^{N} f(x_k^*, y_k^*)\Delta A_k$ is called the Riemann sum of $f(x,y)$.*

Definition 4.3 *If above limit exists, f is said to be integrable over R.*

Proposition 4.4 *Let f and g be integrable functions on a rectangular region R. Then, the following properties of double integrals hold:*

(i) Linearity. For constants a, $b \in \mathbb{R}$,

$$\int\int_R (af(x,y) + bg(x,y))\,dA = a\int\int_R f(x,y)\,dA + b\int\int_R g(x,y)\,dA\,.$$

(ii) If $f(x,y) \geq g(x,y)$ on R,

$$\int\int_R f(x,y)\,dA \geq \int\int_R g(x,y)\,dA\,.$$

(iii) If R is subdivided on two rectangles R_1 and R_2 whose intersection is just a line,

$$\int\int_R f(x,y)\,dA = \int\int_{R_1} f(x,y)\,dA + \int\int_{R_2} f(x,y)\,dA\,.$$

(iv) Let $R = [a,b] \times [c,d]$ and assume $f(x,y) = f_1(x)f_2(y)$, then

$$\int\int_R f(x,y)\,dA = \left(\int_a^b f_1(x)\,dx\right)\left(\int_c^d f_2(y)\,dy\right)\,.$$

Remark 4.5 *The integral $\int\int_R f(x,y)dA$ represents the volume of the region bounded between the xy-plane and the graph of $f(x,y)$.*

Theorem 4.6 (Fubini) *Let $f(x,y)$ be an integrable function over a rectangular region $R = [a,b] \times [c,d]$. Then,*

$$\int\int_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dydx = \int_c^d \int_a^b f(x,y)\, dxdy\,.$$

Example 4.7 *Compute the following double integrals:*

1. $$\int_0^1 \int_{-1}^3 (3 - x + 4y)\, dydx$$

2. *Let* $R = [1,2] \times [0,1]$, $$\int\int_R x^2 y^5\, dA$$

3. *Let* $R = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \le x \le 1, 0 \le y \le \log 5\}$, $$\int\int_R x e^{xy} dA$$

4. $$\int_0^1 \int_0^{\pi/2} x \cos(xy)\, dxdy$$

4.2 Double Integrals Over General Regions

Definition 4.8 *Let $\mathcal{D} \subset \mathbb{R}^2$ be an arbitrary bounded domain and consider a rectangle R such that $\mathcal{D} \subset R$. Assume f is continuous on $\mathcal{D}$ and define*

$$F(x,y) = \begin{cases} f(x,y)\,, & (x,y) \in \mathcal{D}\,, \\ 0\,, & \textit{otherwise}\,. \end{cases}$$

Then,

$$\int\int_{\mathcal{D}} f(x,y)\, dA = \int\int_R F(x,y)\, dA\,.$$

Remark 4.9 *We will consider regions of the form*

$$\mathcal{D}_1 = \{(x,y) \in \mathbb{R}^2 \,|\, a \le x \le b, g_1(x) \le y \le g_2(x)\}\,,$$

and

$$\mathcal{D}_2 = \{(x,y) \in \mathbb{R}^2 \,|\, h_1(y) \le x \le h_2(y), c \le y \le d\}\,.$$

Theorem 4.10 (Fubini) *Let $f(x,y)$ be continuous on $\mathcal{D}_1$ (and $\mathcal{D}_2$), then*

$$\int\int_{\mathcal{D}_1} f(x,y)\,dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx\,,$$

and

$$\int\int_{\mathcal{D}_2} f(x,y)\,dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy\,.$$

Example 4.11 *Compute the following double integrals:*

1. $$\int\int_{\mathcal{D}} x^2 e^{xy}\,dA\,,$$

   where $\mathcal{D}$ is the triangular region enclosed by the lines $y = 1$, $x = 0$ and $y = x/2$.

2. $$\int_0^1 \int_{x^2}^{\sqrt{x}} 160xy^2\,dydx\,.$$

3. $$\int\int_{\mathcal{D}} (x+y)\,dA\,,$$

   where $\mathcal{D}$ is the triangular region enclosed by the lines $y = 0$, $y = 2x$ and $x = 1$.

4. $$\int_0^1 \int_x^1 e^{y^2}\,dydx\,.$$

5. $$\int\int_{\mathcal{D}} (x^2+y^2)\,dA\,,$$

   where $\mathcal{D}$ is the planar region enclosed by the curves $y = 2x$ and $y = x^2$.

6. *Change the order of integration for*

   $$\int_0^2 \int_1^{e^x} f(x,y)\,dydx\,.$$

Definition 4.12 *Given a region $\mathcal{D} \subset \mathbb{R}^2$, the area of $\mathcal{D}$ is the integral*

$$\mathcal{A}(\mathcal{D}) = \int\int_{\mathcal{D}} 1\,dA\,.$$

Example 4.13 *Compute the areas of the following domains $\mathcal{D}$:*

1. *Find the area of the region $\mathcal{D}$ bounded by the parabola $y = x^2 - 2$ and the line $y = x$.*
2. *Area of $\mathcal{D}$ bounded by $y = x^3$ and $y = x^3 + 1$ for $x \in [0,1]$.*
3. *Area of $\mathcal{D}$ bounded by $y = 1$, $y = x$, $y = \log x$ and $y = 0$.*

4.3 Change of Variable in Double Integrals

Definition 4.14 *A transformation $T : \mathcal{D} \subset \mathbb{R}^2 \to \widetilde{\mathcal{D}} \subset \mathbb{R}^2$ defined by $T(u,v) = (x(u,v), y(u,v))$ is bijective (or, one-to-one) if:*

(i) Every point in $\widetilde{\mathcal{D}}$ is in the range of T.

(ii) Different points of $\mathcal{D}$ map to different points in $\widetilde{\mathcal{D}}$.

Theorem 4.15 (Change of Variables) *Let $T(u,v) = (x(u,v), y(u,v))$ be a bijective $\mathcal{C}^1$ transformation such that $T(S) = R$. Let $f(x,y)$ be a continuous and bounded function, then*

$$\int\int_R f(x,y)\, dA = \int\int_S f\left(x(u,v), y(u,v)\right) |\mathrm{detJac}(T)|\, dudv\,,$$

where

$$\mathrm{Jac}(T) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\,,$$

is the Jacobian matrix associated to the transformation.

Remark 4.16 (Polar Coordinates) *We can use the polar coordinates as one possible change of variable:*

$$\begin{aligned} x &= r\cos\theta\,, \\ y &= r\sin\theta\,. \end{aligned}$$

In this case, $|\mathrm{detJac}(T)| = r$.

Example 4.17 *Compute the following integrals using a suitable change of variable:*

1. $$\int_0^1 \int_0^{\sqrt{1-x^2}} \left(x^2 + y^2\right) dydx\,.$$

2. *Let $R = \{(x,y) \in \mathbb{R}^2 \,|\, 1 \le x^2 + y^2 \le 4, x \ge 0\}$ and*
$$\int\int_R (x+y)\, dA\,.$$

3. $$\int_{-\infty}^{\infty} e^{-x^2}\, dx\,.$$

4. *Let R be the parallelogram with vertices $(1,2)$, $(3,4)$, $(4,3)$ and $(6,5)$, and*
$$\int\int_R (x-y)\, dA\,.$$

4.4 Triple Integrals Over General Regions

Theorem 4.18 *Let $f(x,y,z)$ be a continuous and bounded function over the region*

$$B = \{(x,y,z) \in \mathbb{R}^3 \,|\, (x,y) \in \mathcal{D}, u_1(x,y) \leq z \leq u_2(x,y)\}\,.$$

Then,

$$\int\int\int_B f(x,y,z)\,dV = \int\int_{\mathcal{D}} \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA\,.$$

Remark 4.19 *Once we solve the integral with respect to z, we have a double integral over the region $\mathcal{D}$, which we know how to deal with.*

Example 4.20 *Compute the following triple integrals:*

1. $$\int_{-1}^{2}\int_{1}^{2}\int_{0}^{1} z^2 y e^x \,dxdydz\,.$$

2. *Find the volume of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.*

3. *Let B be the region defined by $x^2 + y^2 \leq 4$, $z \geq 0$ and $z \leq 4 - 2y$. Compute*

$$\int\int\int_B x\,dV\,.$$

Remark 4.21 *Change of variables also apply for triple integrals in a similar way as in double integrals.*

Remark 4.22 (Cylindrical Coordinates) *We can use the cylindrical coordinates as one possible change of variable:*

$$\begin{aligned} x &= r\cos\theta\,, \\ y &= r\sin\theta\,, \\ z &= z\,. \end{aligned}$$

In this case, $|\mathrm{detJac}(T)| = r$.

Example 4.23 *Compute the following integrals using cylindrical coordinates:*

1. *Let B be the region bounded by $z = x^2 + y^2$ and $z = 4$. Compute*

$$\int\int\int_B \sqrt{x^2+y^2}\,dV\,.$$

2. *Find the volume of the region B bounded by the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = 2 - x^2 - y^2$.*

Remark 4.24 (Spherical Coordinates) *We can use the spherical coordinates as one possible change of variable:*

$$\begin{aligned} x &= \rho \cos\theta \sin\phi \,, \\ y &= \rho \sin\theta \sin\phi \,, \\ z &= \rho \cos\phi \,. \end{aligned}$$

In this case, $|\mathrm{detJac}(T)| = \rho^2 \sin\phi$.

Example 4.25 *Compute the volume of the region bounded by the cone $z = \sqrt{3\,(x^2 + y^2)}$ and the half-sphere $z = \sqrt{4 - x^2 - y^2}$*

4.5 Exercises

1. $$\int_0^1 \int_0^{\sqrt{1-x^2}} \left(x^2 + y^2\right) dydx \,.$$

   Solution: $\pi/8$.

2. Let $\mathcal{D} = \{(x,y) \in \mathbb{R}^2 \,|\, (x-1)^2 + y^2 \leq 1\}$ and

   $$\int\int_{\mathcal{D}} \left(4 - x^2 - y^2\right) dA \,.$$

   Solution: $5\pi/2$.

3. Let $\mathcal{D}$ be the region bounded by the lines $x+y=1$, $x+y=3$ and the curves $x^2-y^2=-1$, $x^2 - y^2 = 1$. Compute,

   $$\int\int_{\mathcal{D}} (x-y)e^{x^2-y^2} dA \,.$$

   Hint: Use the change of variable $u = x + y$ and $v = x - y$.
   Solution: $2/(3e)$.

4. Let B be the solid bounded by the surface $y^2 = x$, $x = 4$, $y = 0$ and $z = 1$. Compute,

   $$\int\int\int_B xyz \, dV \,.$$

   Solution: 1.

5. Let $B = \{(x,y,z) \in \mathbb{R}^3 \,|\, -y \leq x \leq y, 0 \leq y \leq 1, 0 \leq z \leq 1 - x^4 - y^4\}$. Compute,

   $$\int\int\int_B z \, dV \,.$$

   Solution: $11/18$.

6. Let $B = \{(x, y, z) \in \mathbb{R}^3 \,|\, 1 \leq x^2 + y^2 \leq 9, y \leq 0, 0 \leq z \leq 1\}$. Compute,

$$\int\int\int_B \sqrt{x^2 + y^2}\, dV \,.$$

Solution: $26\pi/3$.

7. Let B be the region bounded by $x^2 + y^2 + z^2 = 16$, $z \geq 0$ and $3z^2 = x^2 + y^2$. Compute,

$$\int\int\int_B z\, dV \,.$$

Solution: 16π.

8. Write the integral for the volume of the region bounded by $z = \sqrt{x^2 + y^2}$ and $z = \sqrt{16 - x^2 - y^2}$, in Cartesian ($x, y$ and z), cylindrical and spherical coordinates. Compute the volume using cylindrical coordinates.
Solution: $64\pi\left(2 - \sqrt{2}\right)/3$.

5 Vector Analysis (Chapter 13)

5.1 Scalar Line Integrals

Remark 5.1 *A scalar line integral is a single integral of a function of several variables along a curve.*

Theorem 5.2 *Let $f(x, y, z)$ be a continuous function defined on a domain that includes the curve C parameterized by $\vec{r}(t)$ with $t \in [a, b]$. Then,*

$$\int_C f\, ds = \int_a^b f\left(\vec{r}(t)\right) \|\vec{r}\,'(t)\|\, dt\,.$$

Remark 5.3 *The scalar line integral is independent of the parameterization of the curve.*

Example 5.4 *Evaluate the following line integrals:*

1. *Evaluate $\int_C (x^2 + yz)\, ds$ where C is parameterized by $\vec{r}(t) = \langle 2t, 5t, -t\rangle$ with $0 \leq t \leq 10$.*
2. *Evaluate $\int_C xe^y\, ds$ where C is the curve of equation $x = e^y$ from $(1, 0)$ to $(e, 1)$.*

Example 5.5 *Find the parameterization of the intersection between $x^2 + y^2 + z^2 = 50$ and the plane $y = 5$.*

Remark 5.6 *If the curve can be divided into different parts (i.e., $C = C_1 \cup C_2$), then*

$$\int_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds\,.$$

Example 5.7 *Evaluate $\int_C xy^2\, ds$ where C is the triangle of vertices $(0, 1, 2)$, $(0, -1, 0)$ and $(1, 0, 3)$.*

5.2 Vector Fields

Definition 5.8 *A vector field (in space) is an assignment of a three-dimensional vector $\vec{F}(x, y, z)$ to each point $(x, y, z) \in \mathcal{D} \subset \mathbb{R}^3$. The subset $\mathcal{D}$ is the domain of the vector field.*

Remark 5.9 *A vector field is a vector-valued function in several variables, i.e., $\vec{F} : \mathcal{D} \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$.*

Example 5.10 $\vec{F}(x, y) = x\vec{i} + y\vec{j}$.

Definition 5.11 *We say that $\vec{F} : \mathcal{D} \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a unit vector field if $\|\vec{F}(x, y, z)\| = 1$ for every $(x, y, z) \in \mathcal{D}$.*

Remark 5.12 *For convenience, we will often use the following notation*

$$\vec{F}(x,y,z) = f_1(x,y,z)\vec{\imath} + f_2(x,y,z)\vec{\jmath} + f_3(x,y,z)\vec{k}\,,$$

where $f_i(x,y,z)$ for $i = 1,2,3$, are real-valued functions in several variables.

Definition 5.13 *We say that a vector field on $\mathbb{R}^3$ is smooth if all derivatives (of all orders) of $f_i(x,y,z)$, $i = 1,2,3$, exist and are continuous in the domain of the vector field.*

Definition 5.14 *Let $\vec{F}(x,y,z) = f_1(x,y,z)\vec{\imath}+f_2(x,y,z)\vec{\jmath}+f_3(x,y,z)\vec{k}$ be a smooth vector field. The divergence of $\vec{F}$ is the function*

$$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\,.$$

Remark 5.15 *We may think ∇ as a formal "vector field"*

$$\nabla = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\,,$$

so the divergence $\nabla \cdot \vec{F}$ is, roughly speaking, the dot product (or scalar/inner) of ∇ and $\vec{F}$.

Remark 5.16 (Physical Interpretation) *If a vector field $\vec{F}$ is the velocity of a fluid moving in space, the divergence at a point measures the flow of the fluid.*

Definition 5.17 *If the divergence of a vector field is zero, we say that the vector field is incompressible.*

Example 5.18 *Compute the divergence of the vector field $\vec{F}(x,y) = x\vec{\imath} + y\vec{\jmath}$.*

Definition 5.19 *Let $\vec{F}(x,y,z) = f_1(x,y,z)\vec{\imath}+f_2(x,y,z)\vec{\jmath}+f_3(x,y,z)\vec{k}$ be a smooth vector field. The curl of $\vec{F}$ is the vector field*

$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix}\,,$$

that is,

$$\nabla \times \vec{F} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\vec{\imath} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\vec{k}\,.$$

Remark 5.20 *As above, understanding ∇ as a "vector field", the curl $\nabla \times \vec{F}$ is the cross product (or vector/outer) of ∇ with $\vec{F}$.*

Remark 5.21 *In the case that the vector field $\vec{F}$ is contained in the plane $\mathbb{R}^2$, then we can consider $f_3 = 0$ and f_1 and f_2 do not depend on z. Therefore,*

$$\nabla \times \vec{F} = \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \vec{k}\,.$$

Remark 5.22 (Physical Interpretation) *The curl of a vector field measures the spin of the vector field, i.e., if $\vec{F}$ is the velocity of a fluid, at each point, the curl measures the tendency of the fluid to rotate. The direction of $\nabla \times \vec{F}$ is parallel to the axis of rotation, while the magnitude represents the speed of rotation.*

Definition 5.23 *We say that a vector field $\vec{F}$ is <u>irrotational</u> if the curl is the vector zero, i.e., if*

$$\nabla \times \vec{F} = \langle 0, 0, 0 \rangle\,.$$

Example 5.24 *Determine if the following vector fields are irrotational or not:*

1. $\vec{F}(x, y) = y\vec{\imath}$.
2. *The gravitational potential energy:*

$$\vec{F}(x, y, z) = \frac{-1}{(x^2 + y^2 + z^2)^{3/2}} \langle x, y, z \rangle\,.$$

Proposition 5.25 *Let $\vec{F}(x, y, z) = f_1(x, y, z)\vec{\imath} + f_2(x, y, z)\vec{\jmath} + f_3(x, y, z)\vec{k}$ be a smooth vector field. Then, the divergence of the curl of $\vec{F}$ is zero, i.e.,*

$$\nabla \cdot \left(\nabla \times \vec{F} \right) = 0\,.$$

Definition 5.26 *Let $f(x, y, z)$ be a smooth function. The <u>gradient</u> vector field of f is*

$$\nabla f = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}\vec{k}\,.$$

Proposition 5.27 *Let $f(x, y, z)$ be a smooth function. Then, the curl of the gradient of f is the vector zero, i.e.,*

$$\nabla \times (\nabla f) = \langle 0, 0, 0 \rangle\,.$$

Remark 5.28 *The other implication is true under an additional assumption.*

Definition 5.29 *We say that a domain $\mathcal{D}$ is <u>connected</u> if any pair of points can be joint by a path contained in $\mathcal{D}$. We say that $\mathcal{D}$ is <u>simply-connected</u> if it is connected and any simple loop in $\mathcal{D}$ can be shrunk to a point.*

Remark 5.30 *In two dimensions, a domain is simply-connected if it is connected and has no holes.*

Proposition 5.31 *Let $\vec{F}$ be a smooth vector field defined on a simply-connected domain. If $\vec{F}$ is irrotational (i.e., $\nabla \times \vec{F} = \langle 0, 0, 0 \rangle$), then $\vec{F}$ is a gradient vector field. That is, it exists a function f such that*

$$\vec{F} = \nabla f \, .$$

Definition 5.32 *The scalar-valued function f is called a (scalar) potential for $\vec{F}$.*

Example 5.33 *Determine if the following vector fields are gradient vector fields and, if yes, find the scalar potentials:*

1. $\vec{F}(x, y) = 2xy^3 \vec{i} + (3x^2y^2 + \cos y)\, \vec{j}$.
2. $\vec{F}(x, y, z) = 2xy\vec{i} + (x^2 + 2yz^3)\, \vec{j} + (3y^2z^2 + 2z)\, \vec{k}$.

Definition 5.34 *Let $f(x, y, z)$ be a smooth function. We define the Laplacian as the function given by the divergence of the gradient of f, i.e.,*

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz} \, .$$

5.3 Vector Line Integrals

Definition 5.35 *An orientation on a curve C is a choice of direction on C. For a closed curve the positive orientation is counter-clockwise, i.e., enclosing a domain on the left.*

Remark 5.36 *Unless the opposite stated, we will always assume that our curves are positively oriented.*

Definition 5.37 *Let $\vec{F}$ be a smooth vector field defined on a domain containing a curve C and assume that C is parameterized by $\vec{r}(t)$ with $t \in [a, b]$. The vector line integral of $\vec{F}$ along C is*

$$\int_C \vec{F} \, ds = \int_a^b \vec{F}\left(\vec{r}(t)\right) \cdot \vec{r}\,'(t) \, dt \, .$$

Remark 5.38 *Sometimes, the vector line integral can also be denoted as*

$$\int_C \vec{F} \, d\vec{r} \, .$$

Example 5.39 *Evaluate the following vector line integrals:*

1. $\int_C \vec{F}\,ds$, *where* $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$ *and* C *is the semicircle parameterized by* $\vec{r}(t) = \langle \cos t, \sin t\rangle$, *with* $0 \le t \le \pi$.

2. $\int_C \vec{F}\,ds$, *where* $\vec{F}(x,y,z) = 4x\vec{i} + 2\vec{j} + 4y^2\vec{k}$ *and* C *is parameterized by* $\vec{r}(t) = \langle 4\cos 2t, 2\sin 2t, 3\rangle$, *with* $t \in [0, \pi/4]$.

3. *The vector line integral of the vector field* $\vec{F}(x,y) = y^2\vec{i} + (2xy+1)\,\vec{j}$ *along the triangle with vertices* $(0,0)$, $(4,0)$ *and* $(0,5)$, *positively oriented.*

Theorem 5.40 (Fundamental Theorem of Line Integrals) *Let* $\vec{F}$ *be a smooth gradient vector field (i.e.,* $\vec{F} = \nabla f$*, for some function* f*) and let* C *be a curve parameterized by* $\vec{r}(t)$ *with* $t \in [a,b]$*. Then,*

$$\int_C \vec{F}\,ds = \int_C \nabla f\,ds = f\left(\vec{r}(b)\right) - f\left(\vec{r}(a)\right).$$

Definition 5.41 *We say that a vector field is* <u>*conservative*</u> *if the line integral* $\int_C \vec{F}\,ds$ *does not depend on the path* C*, but only on the initial and final points.*

Remark 5.42 *If* C *is closed and* $\vec{F}$ *is conservative, then*

$$\int_C \vec{F}\,ds = 0\,,$$

since the initial and final points of a closed curve are the same.

Proposition 5.43 *A vector field is conservative if and only if it is a gradient vector field. In other words, if and only if there exists a scalar potential.*

Remark 5.44 *To sum up, we have that a vector field* $\vec{F}$ *is conservative (i.e., line integrals only depend on the endpoints, not on the path) if and only if* $\vec{F}$ *is a gradient vector field (i.e.,* $\vec{F} = \nabla f$*, for some function* f*). Moreover, gradient vector fields (and so, conservative vector fields) are always irrotational (that is,* $\nabla \times \vec{F} = \nabla \times (\nabla f) = \langle 0,0,0\rangle$*). However, for the converse of this last statement, we need a simply-connected domain.*

Example 5.45 *Consider the vector field*

$$\vec{F}(x,y) = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}\,.$$

Is this vector field irrotational? Can we conclude that it is a gradient vector field? Compute the line integral $\int_C \vec{F}\,ds$ *along the closed circle of radius one centered at* $(0,0)$.

5.4 Green's Theorem

Theorem 5.46 (Green's Theorem) *Let $\mathcal{D}$ be a region in the plane with boundary $\partial\mathcal{D}$ together with the induced positive orientation, and consider a smooth vector field $\vec{F}(x,y) = f_1(x,y)\vec{i} + f_2(x,y)\vec{j}$ defined on $\mathcal{D}$. Then,*

$$\oint_{\partial\mathcal{D}} \vec{F}\, ds = \int\int_{\mathcal{D}} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dA\,.$$

Remark 5.47 *The notation $\oint$ just means that we are computing the line integral $\int$ along closed curves.*

Remark 5.48 *Observe that the integrand on the right-hand side of Green's Theorem is just the curl of $\vec{F}$, $\nabla \times \vec{F}$.*

Example 5.49 *Use Green's Theorem to compute the followings:*

1. *The line integral of the vector field $\vec{F}(x,y) = y^2\vec{i} + (2xy+1)\,\vec{j}$ along the triangle with vertices $(0,0)$, $(4,0)$ and $(0,5)$.*
2. *The line integral of the vector field $\vec{F}(x,y) = \sin(x^2)\vec{i} + (3x-y)\,\vec{j}$ along the triangle with vertices $(-1,2)$, $(4,2)$ and $(4,5)$.*
3. *The area of the region enclosed by the curve $\vec{r}(t) = \langle \sin t \cos t, \sin t\rangle$, $0 \le t \le \pi$.*
4. *The line integral of the vector field $\vec{F}(x,y) = x^3\vec{i} + (5x + e^y \sin y)\,\vec{j}$ along $\partial\mathcal{D}$ where $\mathcal{D} = \{(x,y) \in \mathbb{R}^2 \,|\, 4 \le x^2 + y^2 \le 25\}$.*

Remark 5.50 *The area of a region $\mathcal{D}$ can be computed, for instance, as*

$$\mathcal{A}(\mathcal{D}) = \frac{1}{2}\oint_{\partial\mathcal{D}} \langle -y, x\rangle\, ds\,.$$

There are more ways to compute the area using Green's Theorem.

5.5 Surface Integrals

Remark 5.51 *A surface integral is similar to a line integral, but the integration is over a surface not a curve.*

Definition 5.52 *A parameterized surface is a map $\vec{X}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ from a domain $\mathcal{D} \subset \mathbb{R}^2$ to $\mathbb{R}^3$. The image of $\mathcal{D}$ is the surface $S = \vec{X}(\mathcal{D})$.*

Definition 5.53 *We say that a surface $\vec{X}(u,v)$ is regular if the vector $\vec{X}_u \times \vec{X}_v$ never vanishes. The vector field $\vec{X}_u \times \vec{X}_v$ is the normal vector field to the surface.*

Definition 5.54 *The* <u>*surface integral*</u> *of a function* f *over a surface* S *parameterized by* $\vec{X}(u,v)$ *is*

$$\int\int_S f\,dS = \int\int_{\mathcal{D}} f\left(\vec{X}(u,v)\right) \|\vec{X}_u \times \vec{X}_v\|\,dA\,,$$

where $\mathcal{D}$ *is the domain of the parameterization.*

Definition 5.55 *Let* $\vec{X}(u,v)$ *be a smooth parameterization of a surface* S *with* $(u,v) \in \mathcal{D}$. *The* <u>*surface area*</u> *of* S *is*

$$\mathcal{A}(S) = \int\int_{\mathcal{D}} \|\vec{X}_u \times \vec{X}_v\|\,dA\,.$$

Example 5.56 *Compute the surface area of a sphere of radius* ρ.

Example 5.57 *Find the area of the surface generated by rotating the curve* $y = f(x)$, $x \in (a,b)$, *around the* x*-axis. Answer:*

$$\mathcal{A}(S) = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx\,.$$

Example 5.58 *Calculate the surface integral* $\int\int_S (x+y^2)dS$ *where* S *is the cylinder* $x^2+y^2 = 4$, $0 \le z \le 3$.

Definition 5.59 *An* <u>*orientation*</u> *on a surface* S *is the choice of a unit normal vector* $\vec{N}$ *on* S. *We say that* S *has the* <u>*positive orientation*</u> *if* $\vec{N}$ *points outward (of any convex domain).*

Definition 5.60 *Let* $\vec{F}$ *be a vector field in* $\mathbb{R}^3$ *defined on a domain that contains a surface* S *oriented with a unit normal vector* $\vec{N}$. *The* <u>*flux*</u> *of* $\vec{F}$ *(vector surface integral) along* S *is defined as the surface integral*

$$\int\int_S \vec{F}\,dS = \int\int_S \vec{F}\cdot\vec{N}\,dS = \int\int_{\mathcal{D}} \vec{F}\left(\vec{X}(u,v)\right)\cdot \vec{X}_u \times \vec{X}_v\,dA\,,$$

where $\vec{X}(u,v)$ *is the parameterization of* S.

Example 5.61 *Calculate the flux of the vector field* $\vec{F}(x,y,z) = -y\vec{i} + x\vec{j}$ *over* $\vec{X}(u,v) = \langle u, v^2-u, u+v\rangle$, $0 \le u \le 3$ *and* $0 \le v \le 4$.

5.6 Divergence Theorem

Theorem 5.62 (Divergence Theorem) *Let* B *be a solid in space whose boundary is a surface* S *and assume* S *is positively oriented (i.e., outward unit normal). Then, for a smooth vector field* $\vec{F}$ *defined on* B,

$$\int\int_S \vec{F}\,dS = \int\int\int_B \nabla\cdot\vec{F}\,dV\,.$$

Example 5.63 *Let $\vec{F}(x, y, z) = (x - y)\vec{i} + (x + z)\vec{j} + (z - y)\vec{k}$ and S be the cone $x^2 + y^2 = z^2$, $0 \leq z \leq 1$ with the top disc. Verify the Divergence Theorem.*

Remark 5.64 *The Divergence Theorem allows us to give an interpretation of the divergence vector field. Let B_r be a ball of radius r (very small) centered at P and S_r be the boundary of B_r. The flux of a vector field $\vec{F}$ across S_r can be approximated as*

$$\int\int_{S_r} \vec{F}\, dS = \int\int\int_{B_r} \nabla \cdot \vec{F}\, dV \simeq \nabla \cdot \vec{F}(P)\mathcal{V}(B_r)\,,$$

and we can conclude that

$$\nabla \cdot \vec{F}(P) = \lim_{r \to 0} \frac{1}{\mathcal{V}(B_r)} \int\int_{S_r} \vec{F}\, dS\,.$$

If $\nabla \cdot \vec{F}(P)$ is positive (respectively, negative) it means that the vector field goes outside (resp., inside) the ball.

Example 5.65 *Compute $\int\int_S \vec{F}\, dS$ where S is the cylinder $x^2 + y^2 = 1$, $0 \leq z \leq 2$ and*

$$\vec{F}(x, y, z) = \left(\frac{x^3}{3} + yz\right)\vec{i} + \left(\frac{y^3}{3} - \sin(xz)\right)\vec{j} + (z - x - y)\vec{k}\,.$$

5.7 Stokes' Theorem

Theorem 5.66 (Stokes' Theorem) *Let S be a surface with boundary C, which is simple and closed, positively oriented. Then, for a smooth vector field $\vec{F}$ defined on a domain containing S,*

$$\oint_C \vec{F}\, ds = \int\int_S \nabla \times \vec{F}\, dS\,.$$

Example 5.67 *Let $\vec{F}(x, y, z) = y\vec{i} + 2z\vec{j} + x^2\vec{k}$ and S be $z = 4 - x^2 - y^2$ with $z \geq 0$. Verify Stokes' Theorem.*

Remark 5.68 *If the surface S is a planar region $\mathcal{D}$, Stokes' Theorem is just Green's Theorem.*

Example 5.69 *Compute $\int\int_S \nabla \times \vec{F}\, dS$ where $\vec{F}(x, y, z) = z\vec{i} + x\vec{j} + y\vec{k}$ and for any surface S whose boundary is the circle of radius 1 centered in the origin of the xz-plane.*

5.8 Exercises

1. Compute $\int\int_S z^2\, dS$ where S is the part of the sphere $x^2 + y^2 + z^2 = 4$ above the plane $z = 1$ and the disc enclosed by the intersection of the plane and the sphere.
   Solution: $37\pi/3$.

2. Let $\vec{F}(x,y,z) = xy\vec{i} + (x^2+y^2+z^2)\vec{j} + yz\vec{k}$ be a vector field and C be the boundary of the parallelogram with vertices $(0,0,1)$, $(0,1,0)$, $(2,0,-1)$ and $(2,1,-2)$. Compute $\int_C \vec{F}\,ds$.
   Hint: Use Stokes' Theorem.
   Solution: 3.

3. Consider the vector field $\vec{F}(x,y,z) = x^2y^3\vec{i} + \vec{j} + 2\vec{k}$ and let C be the intersection of the cylinder $x^2+y^2=4$ and the half-sphere $x^2+y^2+z^2=16$, $z \geq 0$. Compute $\int_C \vec{F}\,ds$.
   Solution: -8π.

Review Problems

1. Let $\mathcal{D}$ be the region bounded by the triangle of vertices $(-3,0)$, $(0,0)$ and $(0,3)$ and consider the vector field $\vec{F}(x,y) = (xy^2 + x^2)\,\vec{\imath} + (4x-1)\,\vec{\jmath}$.
   (a) Compute the (vector) line integral of $\vec{F}$ along the triangle $\partial\mathcal{D}$, using the definition of line integrals.
   (b) Compute the (vector) line integral of $\vec{F}$ along the triangle $\partial\mathcal{D}$, using Green's Theorem.

2. Let $\mathcal{D}$ be the region below $x^2 + y^2 = 2$ and above $y = x^2$.
   (a) Compute the area of $\mathcal{D}$ using double integrals.
   (b) Compute the area of $\mathcal{D}$ applying Green's Theorem.
   (c) Compute the line integral of $\vec{F}(x,y) = (x^2 - y)\vec{\imath} + (x - \sin^2 y)\vec{\jmath}$ along the positively oriented boundary of $\mathcal{D}$.

3. Let $\mathcal{D}$ be the region enclosed by $x^2 + y^2 = 25$ and such that $x \leq 0$. Consider the vector field $\vec{F}(x,y) = yx^2\vec{\imath} - x^2\vec{\jmath}$.
   (a) Compute the (vector) line integral of $\vec{F}$ along the boundary $\partial\mathcal{D}$, using the definition of line integrals.
   (b) Compute the (vector) line integral of $\vec{F}$ along the boundary $\partial\mathcal{D}$, using Green's Theorem.

4. Consider the vector field $\vec{F}(x,y,z) = (-4x + 4 - 4y^2 + 2z^3)\,\vec{\imath} + (-8xy + 2z)\,\vec{\jmath} + (6xz^2 + 2y)\vec{k}$.
   (a) Is the vector field $\vec{F}$ conservative?
   (b) If possible, find all the scalar potentials.
   (c) Compute the (vector) line integral of $\vec{F}$ over the closed curve C which is the intersection of the cylinder $x^2 + y^2 = 2x$ and the sphere $x^2 + y^2 + z^2 = 4$.
   (d) Compute the (vector) line integral of $\vec{F}$ over the curve C parameterized by $\vec{r}(t) = \langle t^3 - t^2, \sin^2(\pi t) + t, 1 - \cos(\pi t)\rangle$, for $t \in [0,1]$.

5. Consider the vector field $\vec{F}(x,y,z) = \left(2x + \frac{1}{x} + ye^{xy}\right)\vec{\imath} + (z + xe^{xy})\,\vec{\jmath} + y\vec{k}$.
   (a) Is the vector field $\vec{F}$ conservative?
   (b) If possible, find all the scalar potentials.
   (c) Compute the (vector) line integral of $\vec{F}$ along every possible closed curve in the space.
   (d) Compute the (vector) line integral of $\vec{F}$ along every possible curve joining the points $(1,0,0)$ and $(2,0,1)$.

6. Let B be the region enclosed by $z = x^2 + y^2$ and the plane $z = 9$.

   (a) Compute the volume of B, using triple integrals.

   (b) Compute the volume of B, using the Divergence Theorem.

   (c) Compute the flux of the vector field $\vec{F}(x, y, z) = -y\vec{i} + x\vec{j} + e^{\sqrt{z}}\vec{k}$ over the boundary ∂B.

7. Compute the flux of the vector field $\vec{F}(x, y, z) = yx^2\vec{i} + (xy^2 - 3z^4)\vec{j} + (x^3 + y^2)\vec{k}$ over the boundary of the region enclosed by the sphere of radius 4, $z \leq 0$ and $y \leq 0$.

8. Compute the flux of the vector field $\vec{F}(x, y, z) = \vec{i} + z\vec{j} + 6x\vec{k}$ over the sphere of radius 3.