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Criticality of Sub-Riemannian Geodesics Projections and Applications

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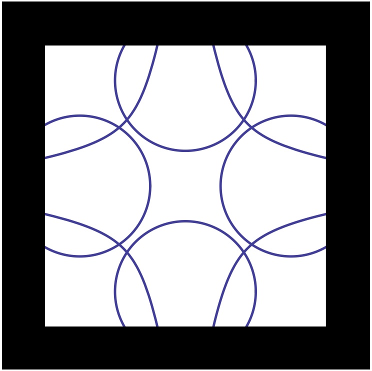

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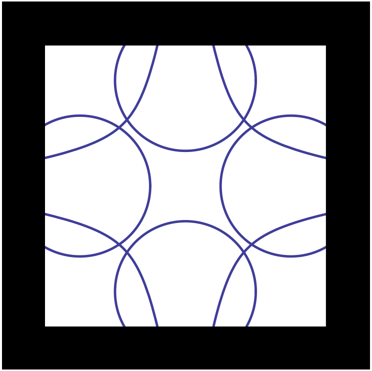

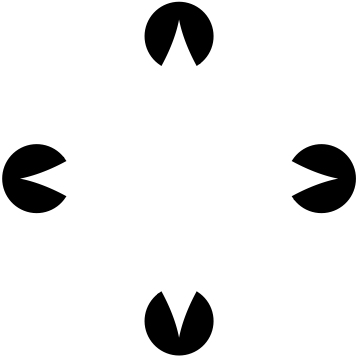

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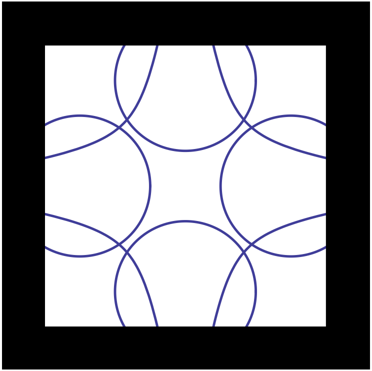

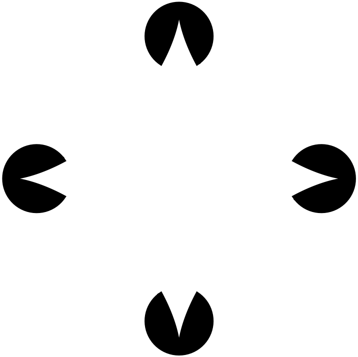

In our brain, the primary visual cortex, $V1$, gives us an intuitive answer.

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- The vector $(\cos\theta, \sin\theta)$ is the direction of maximal rate of change of brightness at point (x, y) of the picture seen by the eye.
- When the cortex cells are stimulated by an image, the border of the image gives a curve inside the space $\mathbb{R}^2 \times \mathbb{S}^1$, but restricted to be tangent to a specific distribution.

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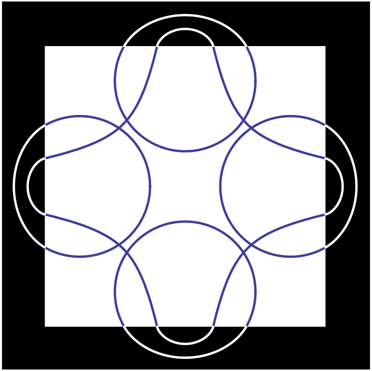
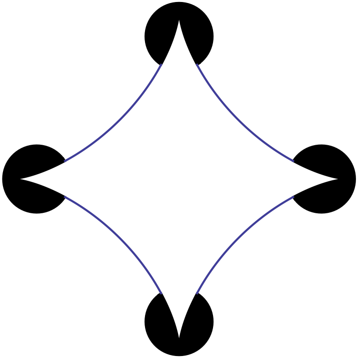
Visual Curve Completion ([3] and [4])

If a piece of the contour of a picture is missing to the eye vision (or maybe it is covered by an object), then the brain tends to complete the curve by minimizing some kind of energy.

Direct Approach to Minimize Length





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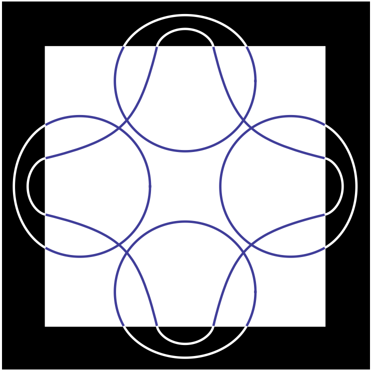

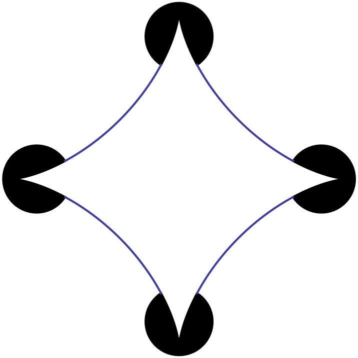

XEL-platform [2] (www.ikergeometry.org)

A gradient descent method useful for an ample family of functionals defined on certain spaces of curves satisfying both affine and isoperimetric constraints.

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Criticality of Projections ([2], [3] and [4])

Geodesics in M^3 are obtained by lifting minimizers (more generally, critical curves) in $\mathbb{R}^2$ of

$$\mathcal{F}(\alpha) = \int_\alpha \sqrt{1 + \kappa^2(s)}\, ds \,.$$

Total Curvature Type Energy

As biological researches suggest, by the hypercolumnar organization of the visual cortex, it may be more accurate to consider the functional

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- If $a = 0$ we get the Total Curvature Functional, and therefore we call $\mathcal{F}$ a total curvature type energy.
- From now on, we are going to consider that $a \neq 0$.

Curvatures of Critical Curves

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As $a \neq 0$, we get the first integral of the Euler-Lagrange Equation,

$$\kappa_s^2 = \left(\frac{\kappa^2+a^2}{a^2}\right)^2\left(d\kappa^2 + a^2(d-a^2)\right).$$

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Thus, we have that the curvature is given by,

$$\kappa(s)=\frac{a\sqrt{d-a^2}\,f\,(as)}{\sqrt{a^2-(d-a^2)\,f^2\,(as)}},$$

where, $f(x)=\sinh x$, $\cosh x$ or e^x.

Different Types of Critical Curves (i)

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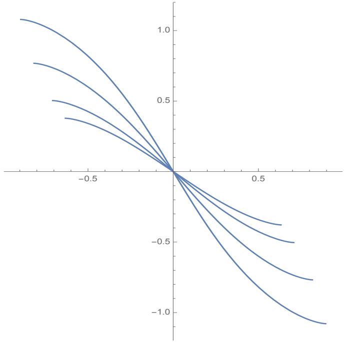
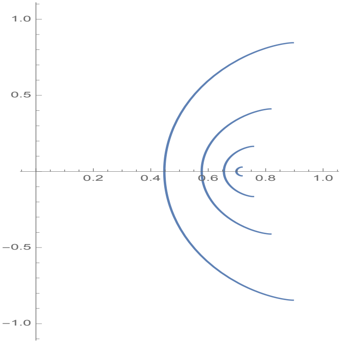
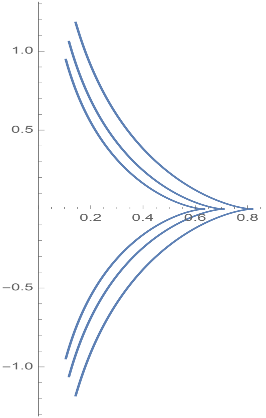
On the left, $f(x) = \sinh x$; and, on the right, $f(x) = \cosh x$,





Different Types of Critical Curves (II)

Finally, here we plot the case $f(x) = e^x$,



Associated Killing Vector Fields

A vector field W along α, which infinitesimally preserves unit speed parametrization is said to be a Killing vector field along α if it evolves in the direction of W without changing shape, only position. That is, if the following equations hold

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- Killing vector fields along curves have unique extensions that are Killing vector fields on the whole space, $\mathbb{R}^3$.

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Let α be a critical curve, then, the binormal evolution surface with initial condition α is a rotational surface.

- S_α has constant negative Gaussian curvature.

Theorem [5]

Let α be a critical curve, then, the binormal evolution surface generated by α verifies $K = -a^2$, where K denotes its Gaussian curvature.

Rotational Surfaces with $K = -a^2$ (I)

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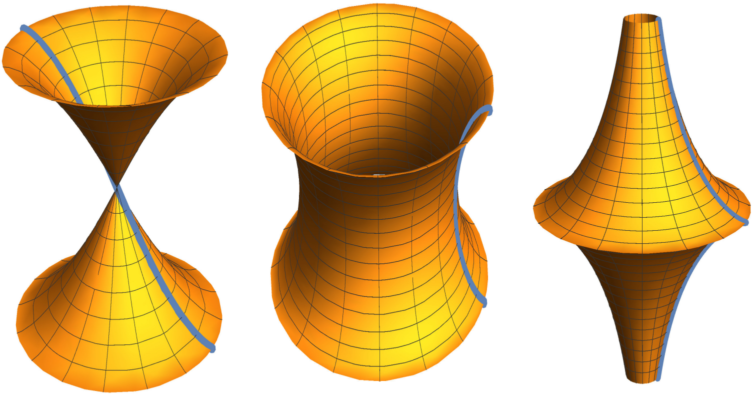
Let M be a rotational surface verifying $K = -a^2$ and let $\gamma(s)$ be its profile curve. Then, γ is a critical curve of the total curvature type energy, $\mathcal{F}$.

Rotational Surfaces with $K = -a^2$ (II)



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Therefore, following this model

- Mechanism of $V1$ may give extra information. That is, not only the completion curve, but also a surface (negative constant Gaussian rotational surface).

References

1. J. Arroyo, O. J. Garay and A. Pámpano, Binormal Motion of Curves with Constant Torsion in 3-Spaces, *Adv. Math. Phys.*, **2017** (2017).
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5. O. J. Garay and A. Pámpano, A Variational Characterization of Profile Curves of Invariant Linear Weingarten Surfaces, *preprint*, (2018).

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THE END

Thank You!

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