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Binormal Evolution of Blaschke's Curvature Energy Extremals in the Minkowski 3-Space

Álvaro Pámpano

Abstract

In 1930, in [4], Blaschke studied the solutions of the variational problem for the energy $\mathbf{\Theta}(\gamma) = \int_\gamma \sqrt{\kappa}$ acting on certain spaces of curves in the Euclidean 3-space $\mathbb{R}^3$. In particular, in $\mathbb{R}^2$, he obtained the catenaries.

In this paper, for a fixed $\mu \in \mathbb{R}$, we are going to extend this problem and we will consider curves in $\mathbb{L}^3$ which are extremals for the action

$$\mathbf{\Theta}(\gamma) = \int_\gamma \sqrt{\kappa - \mu}\,. \tag{1}$$

We are going to get all solutions of the Euler-Lagrange equations of action (1) in Minkowski 3-space $\mathbb{L}^3$, [2].

Finally, making critical curves evolve under their associated Killing vector field ([1] and [6]), these solutions are going to be related with profile curves of constant mean curvature invariant surfaces of $\mathbb{L}^3$; showing that a invariant surface of $\mathbb{L}^3$ has constant mean curvature, if and only if, it is geodesically foliated by critical curves of (1), [2]. This leads to another description of the well-known families of constant mean curvature surfaces in $\mathbb{L}^3$, ([9] and [10]).

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Álvaro Pámpano, *alvaro.pampano@ehu.eus*
Department of Mathematics, Faculty of Science and Technology, Univ. of the Basque Country, Bilbao, Spain

1 Introduction

These notes are a printed version of the talk given by the author at the meeting "Young Researcher Workshop on Differential Geometry in Minkowski Space" held in Granada in April 2017. The purpose of the talk was to offer a partial announcement of some results included in the work [2]. Here, ideas and arguments are only sketched while proofs are omitted. Interested readers are referred to [2] for a complete and more general treatment.

Our background space is going to be the Minkowski 3-space, $\mathbb{L}^3$. That is, $\mathbb{R}^3$ endowed with the canonical metric of index one,

$$g = dx^2 + dy^2 - dz^2 \,,$$

which will be denoted by $\langle , \rangle$. On the other hand, its associated Levi-Civita connection is denoted by $\widetilde{\nabla}$. For more details about this space see, [5].

If $\gamma : I \to \mathbb{L}^3$ is a smooth immersed curve in $\mathbb{L}^3$, $\dot{\gamma}(t)$ will represent its velocity vector $\frac{d\gamma(t)}{dt}$ and the covariant derivative of a vector field $X(t)$ along γ will be denoted by $\frac{DX(t)}{dt}$. A C^1 immersed curve in the Minkowski 3-Space is *spacelike* (respectively, *timelike*; respectively, *lightlike*) if $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle > 0, \forall t \in I$ (respectively, $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle < 0, \forall t \in I$; respectively, $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 0, \forall t \in I$). Of course, there exist curves whose causal character changes as t moves along the parameter interval, but this kind of curves will not be considered here. A non-null curve can be parametrized by the arc-length (along this paper it will be denoted by s) and this natural parameter is called *proper time*.

For a non-null immersed curve, the *first Frenet curvature*, or simply, *the curvature,* is defined as $\kappa = \sqrt{\varepsilon_2 \langle \frac{D\dot{\gamma}(s)}{ds}, \frac{D\dot{\gamma}(s)}{ds} \rangle}$, where ε_2 denotes the causal character of $\frac{D\dot{\gamma}(s)}{ds}$. A geodesic is a constant speed curve whose tangent vector is parallel propagated along itself, i.e. a curve whose tangent, $\dot{\gamma}(s) = T(s)$, satisfies the equation $\frac{DT(s)}{ds} = 0$.

Let γ be a unit speed non-geodesic curve contained in $\mathbb{L}^3$ with non-null velocity $\dot{\gamma} = T$. If it also has non-null acceleration $\frac{D\dot{\gamma}}{ds}$, then γ is a Frenet curve of rank 2 or 3 (see, [7]) and the classical standard Frenet frame along γ is given by $\{T = \dot{\gamma}, N = \frac{\varepsilon_2}{\kappa} \nabla_T T, B\}$, and B is chosen so that $det(T, N, B) = 1$. Then, the *Frenet* *equations* can be written as

$$\begin{aligned}
\frac{DT}{ds} = \widetilde{\nabla}_T T &= \varepsilon_2 \kappa N, \\
\frac{DN}{ds} = \widetilde{\nabla}_T N &= -\varepsilon_1 \kappa T + \varepsilon_3 \tau B, \\
\frac{DB}{ds} = \widetilde{\nabla}_T B &= -\varepsilon_2 \tau N,
\end{aligned}$$

where ε_i, $1 \leq i \leq 3$, denotes the causal character of T, N and B, respectively, and $\{\kappa, \tau\}$ are the *curvature* and *torsion* of γ in $\mathbb{L}^3$. Notice that, even if the rank of γ is 2, the binormal $B = \varepsilon_3 \, T \times N$ is still well defined and Frenet equations still make sense when $\tau = 0$.

Moreover, the fundamental theorem for Frenet curves tells us that, in $\mathbb{L}^3$, the causal character of the Frenet frame and the Frenet curvatures κ, τ completely determine the curve up to isometries. Moreover, given functions κ and τ we can always construct a spacelike (respectively, timelike) Frenet curve, parametrized by the arc-length, whose curvature and torsion are precisely κ and τ.

2 Extension of a Blaschke's Variational Problem

We denote by $\Omega_{p_o p_1}$ the space of smooth immersed curves of $\mathbb{L}^3$ joining two points of it, and verifying that $\kappa - \mu > 0$. We are going to consider the curvature energy functional acting on $\Omega_{p_o p_1}$

$$\mathbf{\Theta}(\gamma) = \int_\gamma \sqrt{\kappa - \mu} = \int_0^L \sqrt{\kappa(s) - \mu} \, ds \,,$$

where $\mu \in \mathbb{R}$ is a fixed real constant. Take into account that $\kappa = \mu$ would be a global minimum if we were considering $L^1([0, L])$ as the space of curves.

On the other hand, working with the space of curves $\Omega_{p_o p_1}$, we will be able to apply the fundamental lemma of the calculus of variations, since for every variation by immersed curves we can always find a subvariation by curves verifying $\kappa - \mu > 0$.

In 1930, in [4], Blaschke studied the case $\mu = 0$ in the Euclidean 3-space. Our curvature energy functional (1) represents an extension of Blaschke's variational problem for the Minkowski space of dimension 3, $\mathbb{L}^3$, and we are interested in studying critical curves in this space.

For this purpose, we are going to obtain the Euler-Lagrange equations for the curvature energy functional $\mathbf{\Theta}(\gamma) = \int_\gamma \sqrt{\kappa - \mu}$, in $\mathbb{L}^3$. These differential equations can be written as

$$\frac{d^2}{ds^2}\left(\frac{\varepsilon_2}{\sqrt{\kappa-\mu}}\right) + \frac{1}{\sqrt{\kappa-\mu}}(\varepsilon_1\kappa^2 - \varepsilon_3\tau^2) = 2\varepsilon_1\kappa\sqrt{\kappa-\mu}\,, \tag{2}$$

$$\frac{d}{ds}\left(\frac{\tau}{\kappa-\mu}\right) = 0\,. \tag{3}$$

Under suitable boundary conditions, solutions of these equations are going to be critical curves for our energy functional.

In what follows we are going to get explicit solutions of the Euler-Lagrange equations in terms of the curvature and torsion of the curve.

Let's consider first that the curvature is constant, $\kappa = \kappa_o \in \mathbb{R}^+$. Then, the second Euler-Lagrange equation (3) implies that the torsion is constant, that is, $\tau = \tau_o \in \mathbb{R}$. Thus, in this case, γ must be a Frenet helix. Moreover, substituting this in the first Euler-Lagrange equation (2) we get the relation between the curvature and torsion of γ

$$\kappa_o = \mu + \sqrt{\mu^2 - \varepsilon_1\varepsilon_3\tau_o^2}.$$

Suppose now that the curvature is not constant, then let's call

$$\begin{aligned} a &= -\varepsilon_1\varepsilon_2\mu^2\,, \\ b &= 4\varepsilon_2 d + 2\varepsilon_1\varepsilon_2\mu\,, \\ c &= -\varepsilon_1\varepsilon_2 - \varepsilon_2\varepsilon_3 e^2\,, \end{aligned}$$

and $\Delta = 4ac - b^2 = -16d^2 - 16\varepsilon_1\mu d + 4\varepsilon_1\varepsilon_3\mu^2 e^2$, where d, e are real constants, constants of integration. Then, calling $x = \kappa - \mu$, the first integrals of the Euler-Lagrange equations (2)-(3) reduce to

$$\begin{aligned} x_s^2 &= 4x^2(cx^2 + bx + a)\,, \\ \tau &= ex\,. \end{aligned}$$

By a simple analysis of the first equation, we realize that the following cases are not possible:

1. $\Delta \geq 0$ and $c < 0$,
2. $a \leq 0$, $2d = -\varepsilon_1\mu$ and $e^2 = -\varepsilon_1\varepsilon_3$.

In the rest of the cases, we can integrate the Euler-Lagrange equations (2)-(3), and get the explicit formulas for the curvature of the critical curves. After long computations, we obtain

1. If $\Delta \neq 0$ and $a \neq 0$

$$\kappa(s) = \frac{2a + \mu(-b + \sqrt{|\Delta|}f(2\mu s))}{-b + \sqrt{|\Delta|}f(2\mu s)}\,,$$

   where, $f(x) = \sinh x$, if $\Delta > 0$ and $a > 0$; $f(x) = \cosh x$, if $\Delta < 0$ and $a > 0$; and $f(x) = \sin x$, if $\Delta < 0$ and $a < 0$.

2. If $\Delta = 0$ and $a > 0$

$$k(s) = \frac{\mu + (2a - b\mu)\exp(2\mu s)}{1 - b\exp(2\mu s)}\,.$$

3. If $\Delta < 0$ and $a = 0$, that is, $(\mu = 0)$

$$k(s) = \frac{b}{-c + b^2 s^2}\,.$$

4. If $\Delta = 0$ and $a = 0$, that is, $(\mu = d = 0)$

$$k(s) = \frac{1}{2\sqrt{cs}}\,.$$

Observe that, in all the cases above, the torsion of γ obtained from (3) is given by

$$\tau = e(\kappa - \mu)\,,$$

where $e \in \mathbb{R}$ is one of the constants of integration. Thus, given the causal characters ε_i of the Frenet frame, by the fundamental theorem of Frenet curves in $\mathbb{L}^3$ we have determined a unique curve up to rigid motions.

3 Extremals of Blaschke's Curvature Energy

In this section, we are going to give a geometric interpretation of extremals of $\mathbf{\Theta}(\gamma) = \int_\gamma \sqrt{\kappa - \mu}\, ds$. Let's start with the planar case. Take $\tau = 0$, then we know that γ must lie in a totally geodesic surface of $\mathbb{L}^3$, that is a Riemannian or Lorentzian plane, since we are working with non-null curves. Suppose first that γ lies in a Riemannian plane, that is we can assume that $\gamma \subset \mathbb{R}^2$, then similarly as in the Euclidean case ([2] and [6]), we have that

Theorem 3.1. *Critical curves of* $\mathbf{\Theta}(\gamma) = \int_\gamma \sqrt{\kappa - \mu}\, ds$ *in* $\mathbb{R}^2$ *are precisely the Delaunay curves, that is, the roulettes of foci of conics (lines, circles, catenaries, nodaries and undularies).*

Moreover, in [8], Hano and Nomizu gave a description of spacelike roulettes in the Minkowski plane $\mathbb{L}^2$, and therefore, if we suppose that γ is contained in $\mathbb{L}^2$ we are able to prove

Theorem 3.2. *The locus of the origin when a part of a spacelike quadratic curve is rolled along a spacelike line is a spacelike critical curve for* $\mathbf{\Theta}(\gamma) = \int_\gamma \sqrt{\kappa - \mu}\, ds$ *in* $\mathbb{L}^2$.

That is, planar extremals of the extended Blaschke's curvature energy represent the spacelike roulettes of foci of conics in $\mathbb{L}^3$.

On the other hand, if the critical curve of the action (1) is not planar we have the following relation between its curvature and torsion

$$e\mu = e\kappa - \tau \,. \tag{4}$$

As we already said in the previous section, if the curvature of γ, κ, is constant then so is its torsion τ and thus, we are dealing with Frenet helices.

Furthermore, when $\mu = 0$, our critical curves are *Lancret curves*, that is, curves making a constant angle with a fixed direction. These curves in $\mathbb{L}^3$ are characterized by $\tau = \lambda\kappa$ for some non-zero constant λ, [3].

Finally, if $\mu \neq 0$, above relation (4) between the curvature and the torsion implies that γ is a *Bertrand curve* (see [11] for the definition and characterization of Bertrand curves in the Minkowski 3-Space) and it can be proved that its Bertrand mate is not critical for our energy functional (1).

4 Binormal Evolution of Extremals

In this part we are going to relate our critical curves with invariant surfaces of constant mean curvature, following the method described in ([1] and [6]). This construction of immersed surfaces consist on letting γ evolve under its associated Killing vector field.

A vector field W along γ, which infinitesimally preserves the parametrization by its proper time is said to be a Killing vector field along γ if it evolves in the direction of W without changing shape, only position. That is, if the following equations hold

$$W(v)(\bar{t},0) = W(\kappa)(\bar{t},0) = W(\tau)(\bar{t},0) = 0\,.$$

Let's consider our functional $\mathbf{\Theta}(\gamma) = \int_\gamma \sqrt{\kappa - \mu}\, ds$ acting on the space $\Omega_{p_o p_1}$ and take γ a solution of the Euler-Lagrange equations, then the vector field

$$\mathcal{I} = \frac{1}{2\sqrt{\kappa - \mu}} B \tag{5}$$

is a Killing vector field along γ, [1], [6].

In our case, since $\mathbb{L}^3$ is complete, we can consider the one-parameter group of isometries determined by the flow of $\mathcal{I}$ (5) that is, $\{\phi_t, t \in \mathbb{R}\}$. Now, we construct the surface $S_\gamma := \{x(s,t) := \phi_t(\gamma(s))\}$ by evolving γ under the flow of $\mathcal{I}$. The surface S_γ is a $\mathcal{I}$-invariant surface whose mean curvature is

$$H = -\varepsilon_1 \varepsilon_2 \mu.$$

Finally, as $\mu \in \mathbb{R}$ is fixed, S_γ has constant mean curvature.

Notice that the immersed surfaces generated by evolving in the direction of the binormal a Bertrand curve are called *Razzaboni surfaces*, [11]. Thus, our $\mathcal{I}$-invariant surfaces of constant mean curvature are Razzaboni surfaces.

Now, for the converse, assume that S is a non-degenerate $\mathcal{G}_\xi$-invariant surface of $\mathbb{L}^3$, i.e., for any $x \in S$ and $\Phi_t \in \mathcal{G}_\xi$ we have $\Phi_t(S) = S$. The following theorem characterizes every constant mean curvature (CMC) ξ-invariant surfaces of $\mathbb{L}^3$,

Theorem 4.1. *Any ξ-invariant CMC surface S of $\mathbb{L}^3$ admits a local geodesic parametrization where the leaves provide a geodesic foliation by critical curves of the extended Blaschke's variational problem with $\mu = -\varepsilon_1\varepsilon_2 H$.*

Thus, every ξ-invariant CMC surface is a ruled surface or, it is generated by evolving a critical curve of $\mathbf{\Theta}(\gamma) = \int_\gamma \sqrt{\kappa + \varepsilon_1\varepsilon_2 H}\, ds$ under the flow of the Killing vector field ξ. This section leads to another description of the well-known families of constant mean curvature surfaces in $\mathbb{L}^3$, (see [9] and [10]).

5 Isometric Families of CMC Surfaces

The method developed in previous sections also allows us to deform CMC surfaces isometrically preserving the mean curvature. In fact, applying Theorem 4.1, we get that a ξ-invariant surface of CMC which is not a plane or a cylinder (both Riemannian or Lorentzian) can isometrically be deformed if the following relations involving the constants d and e,

1. If $\Delta \neq 0$ and $a \neq 0$, $\Delta = \nu b^2$,
2. If $\Delta = 0$ and $a > 0$, (there is no any isometric deformation),
3. If $\Delta < 0$ and $a = 0$, $c = \nu b^2$ and
4. If $\Delta = a = 0$, (as $d = 0$, there is no biparametric family) there is no any constraint,

are verified for some $\nu \in \mathbb{R}$. Then for each correspondent solution of the Euler-Lagrange equations, we can prove

Theorem 5.1. *For each real constant μ, let $\{S_\gamma\}_{d,e}$ be the family of ξ-invariant surfaces shaped on a critical curve γ of $\mathbf{\Theta}(\gamma) = \int_\gamma \sqrt{\kappa - \mu}\, ds$. Under the relations above (except for case 2), the family $\{S_\gamma\}_d$ is generated by isometric surfaces with the same constant mean curvature $H = -\varepsilon_1\varepsilon_2\mu$.*

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