Tutorial on XPPAUT: Numerical Bifurcation Diagrams for ODEs

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April 28, 2015
BioMath Seminar

Outline

- Introduction to XPPAUT
- Getting started
- Plotting solutions to ODEs
- Phase planes and drawing nullclines
- Using AUTO to draw bifurcation diagrams
- Exercise

Introduction to XPPAUT

- XPP is a general numerical tool for simulating, animating, and analyzing dynamical systems
  - Differential equations
  - Delay equations
  - Volterra integral equations
  - Discrete dynamical systems
  - Markov processes
- AUTO is a programs built for bifurcation analysis.
  - AUTO is built into xppaut

Getting started

Download here:
`http://www.math.pitt.edu/~bard/xpp/xpp.html`

Useful tutorial:
`http://www.math.pitt.edu/~bard/bardware/tut/start.html`

Creating an ODE file

- ODE files are ASCII readable files that the XPP parser reads to create machine useable code
- ODE file has the equations, parameters, variables, boundary conditions, and functions for your model.
- The methods of solving the equations and graphics are all done within the program

LVE.ode example

Lotka-Volterra equations

$$\frac{dx}{dt} = bx\left(1 - \frac{x}{K}\right) - \frac{cx}{a+x}y$$
$$\frac{dy}{dt} = \frac{cx}{a+x}y - dy$$

LVE.ode example

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- Hopf bifurcation is a critical point where a system's stability switches and a periodic solution arises
- local bifurcation in which a fixed point of a dynamical system loses stability, as a pair of complex conjugate eigenvalues (of the linearization around the fixed point) cross the complex plane imaginary axis.

LVE.ode example

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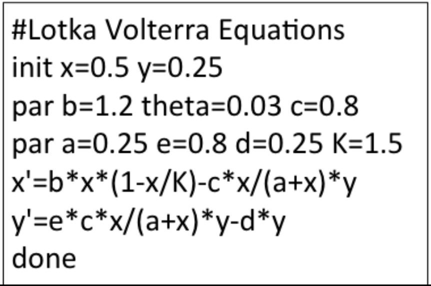
LVE.ode



```
#Lotka Volterra Equations
init x=0.5 y=0.25
par b=1.2 theta=0.03 c=0.8
par a=0.25 e=0.8 d=0.25 K=1.5
x'=b*x*(1-x/K)-c*x/(a+x)*y
y'=e*c*x/(a+x)*y-d*y
done
```

LVE.ode example

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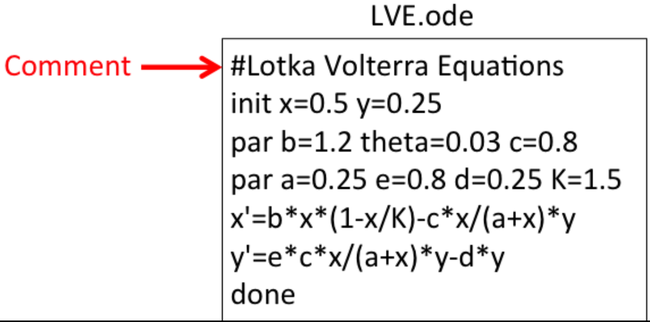

LVE.ode

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LVE.ode example

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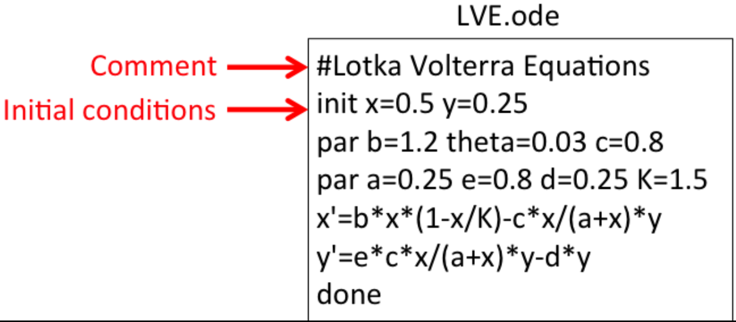

LVE.ode

Comment

Initial conditions

```
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LVE.ode example

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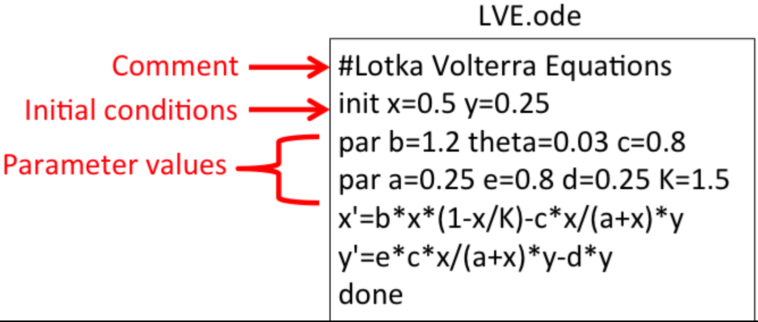

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LVE.ode example

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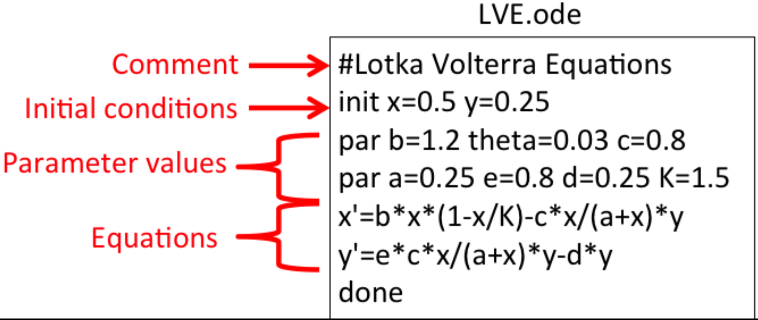

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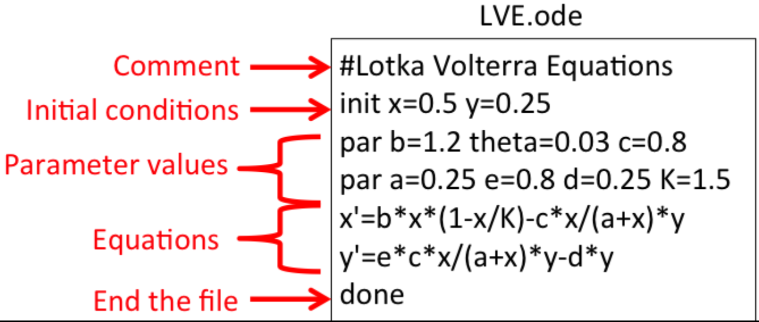

LVE.ode

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done
```
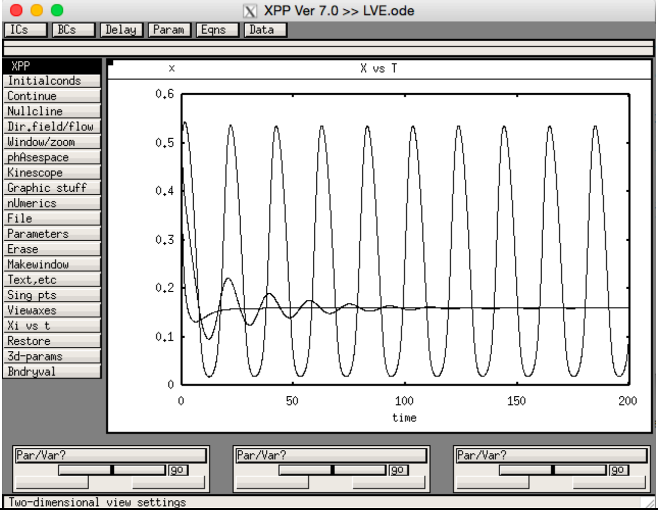

LVE.ode example

→ plot solutions

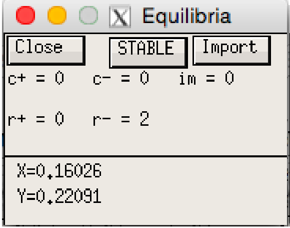


Two-dimensional view settings

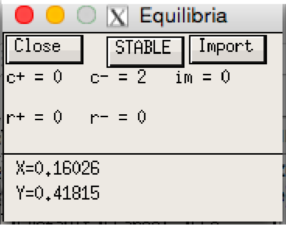
LVE.ode example

→ Equlibria and stability

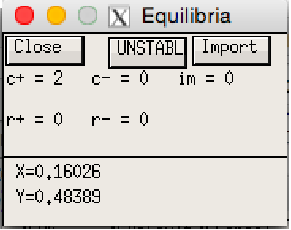
- (S)ing pts: calculate equilibria for a system.
- (G)o begins the calculation using the values in the initial data box as a first guess. Newton's method is applied.
- Can print out eigenvalues and plot stable/unstable manifolds
- Determines stability



```
c+ = 0   c- = 0   im = 0

r+ = 0   r- = 2

X=0.16026
Y=0.22091
```



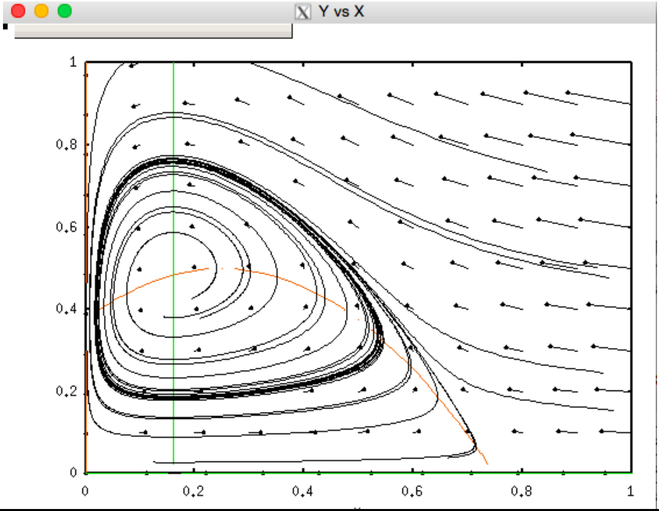
```
c+ = 0   c- = 2   im = 0

r+ = 0   r- = 0

X=0.16026
Y=0.41815
```



```
c+ = 2   c- = 0   im = 0

r+ = 0   r- = 0

X=0.16026
Y=0.48389
```

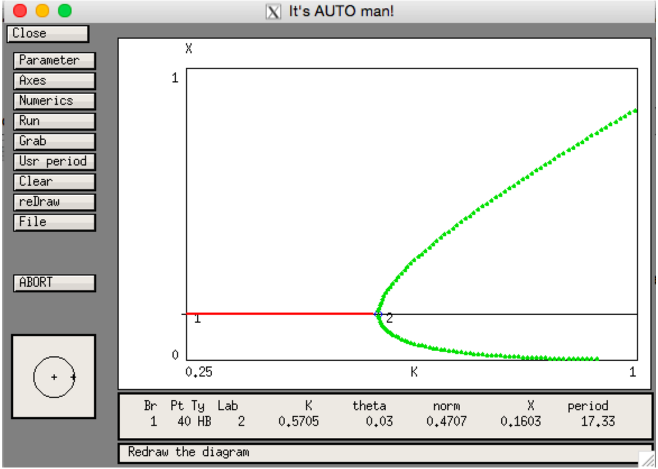
LVE.ode example

→ direction field, nullclines



LVE.ode example

→ bifurcation diagram with AUTO



```
Br  Pt Ty  Lab      K      theta    norm      X     period
 1  40 HB    2  0.5705     0.03   0.4707  0.1603    17.33
```

```
Redraw the diagram
```

MorrisLecar.ode example

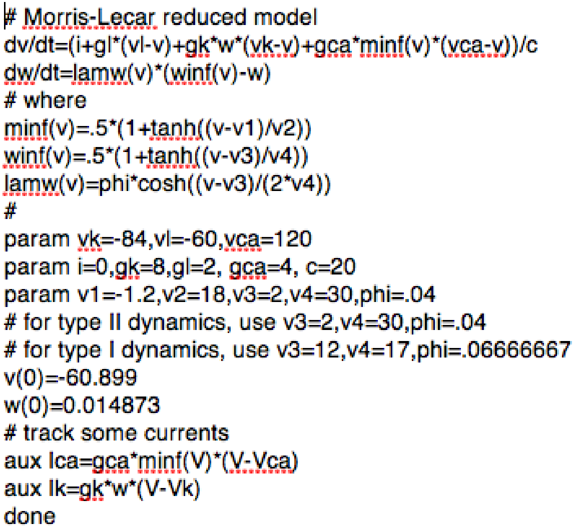
Morris-Lecar equations

$$
\begin{aligned}
C\frac{dV}{dt} &= g_L(V - V_L) - G_{Ca}m_\infty(V)(V - V_{Ca}) - g_K w(V - V_k) + I \\
\frac{dw}{dt} &= \lambda_w(V)(w_\infty(V) - w)
\end{aligned}
$$

where

$$
\begin{aligned}
w_\infty(V) &= \frac{1}{2}(1 + \tanh((V - V_3)/V_4)) \\
\lambda_w(V) &= \phi \cosh((V - V_3)/2V_4))
\end{aligned}
$$

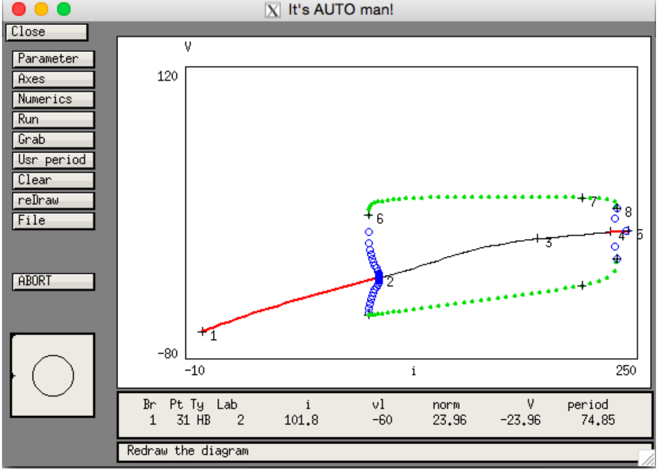
MorrisLecar.ode example



```
# Morris-Lecar reduced model
dv/dt=(i+gl*(vl-v)+gk*w*(vk-v)+gca*minf(v)*(vca-v))/c
dw/dt=lamw(v)*(winf(v)-w)
# where
minf(v)=.5*(1+tanh((v-v1)/v2))
winf(v)=.5*(1+tanh((v-v3)/v4))
lamw(v)=phi*cosh((v-v3)/(2*v4))
#
param vk=-84,vl=-60,vca=120
param i=0,gk=8,gl=2, gca=4, c=20
param v1=-1.2,v2=18,v3=2,v4=30,phi=.04
# for type II dynamics, use v3=2,v4=30,phi=.04
# for type I dynamics, use v3=12,v4=17,phi=.0666667
v(0)=-60.899
w(0)=0.014873
# track some currents
aux Ica=gca*minf(V)*(V-Vca)
aux Ik=gk*w*(V-Vk)
done
```

MorrisLecar.ode example

→ bifurcation diagram with AUTO



```
Br  Pt Ty  Lab        i         vl       norm         V      period
 1  31 HB    2    101.8        -60      23.96     -23.96      74.85
```

```
Redraw the diagram
```

Thank you

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