8 Stochastic Differential Equations

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Lecture notes follow: Allen, Linda JS. An introduction to stochastic processes with applications to biology. CRC Press, 2010.

Definitions and Notation

- Let $\{X(t)\}$ be a collection of continuous random variables defined on a probability space, a stochastic process that is continuous in time $t \in [t_0, \infty)$ and in state

$$X(t) \in (-\infty, \infty) \text{ or } [0, \infty) \text{ or } [0, M]$$

- The probability density function $p(x, t)$ is associated with $X(t)$
- To find a probability associated with $X(t)$ you need to integrate:

$$\text{Prob}\{X(t) \in [a, b]\} = \int_b^a p(x, t) dx$$

Definitions and Notation

Markov Property

Assume $\{X(t) : t \in [0, \infty)\}$ is a stochastic process continuous in time with a continuous state space. It is a Markov process if

$$\begin{aligned}\text{Prob}\{X(t_n) \le y | X(0) &= x_0, X(t_1) = x_1, ..., X(t_{n-1}) = x_{n-1}\} \\ &= \text{Prob}\{X(t_n) \le y | X(t_{n-1}) = x_{n-1}\}\end{aligned}$$

for a given sequence of times $0 < t_0 < t_1 < ...t_n$.

The future state of the process only depends on the current state.

Definitions and Notation

Transition p.d.f

The transision p.d.f $p(y, s; x, t)$ is the density function for a transition from state x at time t to state y at time s, $t < s$.
It is homogeneous if

$$p(y, s + \Delta t; x, t + \Delta t) = p(y, s; x, t)$$

and denoted

$$p(y, x, s - t)$$

the transitions depend only of the length of time between states, $s - t$.

Chapman-Kolmogorov equations

$$p(y, s; x, t) = \int_{-\infty}^{\infty} p(y, s; z, u) p(z, u; x, t) dz$$

where $t < u < s$.

Definitions and Notation

- The dynamics depend on the initial density of $X(0)$
  - usually the initial density is concentrated at x_0
  - this means the p.d.f. of $X(0)$ is a Dirac delta function

$$\delta(x - x_0) = 0, \qquad x \neq x_0, \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x - x_0) dx = 1$$

  - for simplicity we just write $X(0) = x_0$ when the initial p.d.f. is $p(x, t_0) = \delta(x - x_0)$
  - here the p.d.f. of $X(t)$, $p(x, t)$ is the same as the transition probability density function, $p(x, t; x_0, 0)$

Random Walk and Brownian Motion

- Consider a random walk on the set $\{0, \pm\Delta x, \pm 2\Delta x, ...\}$
- Let p be the probability of moving right
- q be the probability of moving left
- $p + q = 1$
- Let $X(t) \in \{0, \pm\Delta x, \pm 2\Delta x, ...\}$ be the DTMC for this random walk, where $t \in \{0, \Delta t, 2\Delta t, ...\}$ and

$$p_x(t) = \text{Prob}\{X(t) = x\} = u(x, t)$$

Random Walk and Brownian Motion

- It follows that

$$u(x, t + \Delta t) = pu(x - \Delta x, t) + qu(x + \Delta x, t)$$

- Expanding the right-hand side using Taylor's formula about the point (x, t) yields

$$\begin{aligned} u(x,t + \Delta t) = \\ & p\left[u(x,t) + \frac{\partial u(x,t)}{\partial x}(-\Delta x) + \frac{\partial^2 u(x,t)}{\partial x^2}\frac{(\Delta x)^2}{2} + o((\Delta x)^3)\right] \\ & + q\left[u(x,t) + \frac{\partial u(x,t)}{\partial x}(\Delta x) + \frac{\partial^2 u(x,t)}{\partial x^2}\frac{(\Delta x)^2}{2} + o((\Delta x)^3)\right] \\ & = u(x,t) + (q-p)\frac{\partial u(x,t)}{\partial x}(\Delta x) + \frac{\partial^2 u(x,t)}{\partial x^2}\frac{(\Delta x)^2}{2} + o((\Delta x)^3) \end{aligned}$$

Random Walk and Brownian Motion

- subtracting $u(x,t)$ and dividing by Δt yields

$$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} = (q-p)\frac{\partial u(x,t)}{\partial x}\frac{\Delta x}{\Delta t}+\frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x^2}\frac{(\Delta x)^2}{\Delta t}+o\left(\frac{(\Delta x)^3}{\Delta t}\right)$$

- Assume

$$\lim_{\Delta t,\Delta x\to 0} = (p-q)\frac{\Delta x}{\Delta t} = c$$

$$\lim_{\Delta t,\Delta x\to 0} = \frac{(\Delta x)^2}{\Delta t} = D$$

$$\lim_{\Delta t,\Delta x\to 0} = \frac{(\Delta x)^3}{\Delta t} = 0$$

- Letting $\Delta t, \Delta x \to 0$, the probability $u(x,t)$ represents the p.d.f of a continuous-time and continuous-state process $X(t)$ which is a solution of the PDE

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}+\frac{D}{2}\frac{\partial^2 u}{\partial x^2}, \qquad x\in(-\infty,\infty)$$

Random Walk and Brownian Motion

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} + \frac{D}{2}\frac{\partial^2 u}{\partial x^2}, \qquad x \in (-\infty, \infty)$$

- This PDE is known as the **diffusion equation with drift**
  - D is the diffusion coefficient
  - c is the drift coefficient
- This PDE is also known as the **forward Kolmogorov differential equation** for this process
- When $p = q = 1/2$ the movement is unbiased and the limiting stochastic process is known as **Brownian motion**:

$$\frac{\partial u}{\partial t} = \frac{D}{2}\frac{\partial^2 u}{\partial x^2}, \qquad x \in (-\infty, \infty)$$

- Standard Brownian motion ($X(0) = 0$ and $D = 1$) is also known as the **Wiener process**.

Random Walk and Brownian Motion

- The assumptions on the limits in the random walk model were necessary to obtain the diffusion equation with drift.
- These assumptions are very important in the derivation of the Kolmogorov differential equations
- they are are related to the infinitesimal mean and variance of the process.

Example

Brownian Motion Example

Consider the equation for Brownian motion with the initial condition $X(0) = x_0$

$$\frac{\partial u}{\partial t} = \frac{D}{2}\frac{\partial^2 u}{\partial x^2}, \qquad x \in (-\infty, \infty)$$

1. Verify that

$$u(x,t) = \frac{1}{\sqrt{2\pi Dt}} \exp\left(-\frac{(x-x_0)^2}{2Dt}\right)$$

   is a solution
2. What kind of distribution does the p.d.f have?
3. The p.d.f has a normal distribution.
4. What is the mean? What is the variance?

Diffusion Process

The diffusion process is a Markov process with additional properties on the infinitesimal mean and variance

Diffusion Process

Let $\{X(t) : t \in [0, \infty)\}$ be a Markov process with state space $(-\infty, \infty)$ having continuous sample paths and trasition p.d.f given by $p(y, s; x, t)$, $t < s$. Then $\{X(t)\}$ is a diffusion process if its p.d.f satisfies the following 3 assumptions for any $\epsilon > 0$ and $x \in (-\infty, \infty)$:

1. $\displaystyle \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \int_{|y-x|>\epsilon} p(y, t + \Delta t; x, t) dy = 0$
2. $\displaystyle \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \int_{|y-x|\leq\epsilon} (y - x) p(y, t + \Delta t; x, t) dy = a(x, t)$
3. $\displaystyle \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \int_{|y-x|\leq\epsilon} (y - x)^2 p(y, t + \Delta t; x, t) dy = b(x, t)$

Here $a(x, t)$ is the drift coefficient and $b(x, t)$ is the diffusion coefficient.

Diffusion Process

- Similar but slightly stronger conditions that lead to the conditions above are expressed in terms of the expectation:
  1. $\lim_{\Delta t \to 0^+} \frac{1}{\Delta t} E(|\Delta X(t)|^\delta | X(t) = x) = 0, \quad \delta > 2$
  2. $\lim_{\Delta t \to 0^+} \frac{1}{\Delta t} E(\Delta X(t) | X(t) = x) = a(x,t)$
  3. $\lim_{\Delta t \to 0^+} \frac{1}{\Delta t} E([\Delta X(t)]^2 | X(t) = x) = b(x,t)$
- where $\Delta x(t) = X(t + \Delta t) - X(t) = y - x$
- Here $a(x,t)$ is the drift coefficient
  - the expected change in a small increment of X starting at x
- $b(x,t)$ is the diffusion coefficient
  - the variance in a small increment of X starting at x

Kolmogorov Differential Equations

- The Forward and Backward Kolmogorov Differential Equations follow from these assumptions
- The **backward Kolmogorov DE** for a time-homogeneous process is

$$\frac{\partial p(y,x,t)}{\partial t} = a(x)\frac{\partial p(y,x,t)}{\partial x} + \frac{1}{2}b(x)\frac{\partial^2 p(y,x,t)}{\partial x^2}$$

- The **forward Kolmogorov DE** for a time-homogeneous process is

$$\frac{\partial p(y,x,t)}{\partial t} = -\frac{\partial [a(y)p(y,x,t)]}{\partial y} + \frac{1}{2}\frac{\partial^2 [b(y)p(y,x,t)]}{\partial y^2}$$

- The p.d.f. $p(x,t)$ with $p(x,0) = \delta(x - x_0)$ is a solution of the forward Kolmogorov DE, therefore we can replace $p(y,x,t)$ with $p(x,t)$:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial [a(x)p(x,t)]}{\partial x} + \frac{1}{2}\frac{\partial^2 [b(x)p(x,t)]}{\partial x^2}$$

Wiener Process

- Wiener process is a continuous-time stochastic process named in honor of Norbert Wiener.
- It is often called standard Brownian motion due to its historical connection with the physical process known as Brownian motion originally observed by Robert Brown.
- Suppose $W(t)$ is the displacement of a small particle from the origin.
- The displacement of the particle over the time interval t_1 to t_2 is long compared to the time between impacts.
- The central limit theorem can be applied to the sum of a large number of these small disturbances so that $W(t_2) - W(t_1)$ has a normal density.

Wiener Process

Wiener Process

The stochastic process $\{W(t) : t \in [0, \infty)\}$ is a Wierner process if $W(t) \in (-\infty, \infty)$ depends continuously on t and the following conditions hold:

1. For $0 \leq t_1 \leq t_2 \leq \infty$, $W(t_2) - W(t_1)$ is normally distributed with mean 0 and variance $t_2 - t_1$.
   - $W(t_2) - W(t_1) \sim N(0, t_2 - t_1)$
2. For $0 \leq t_0 \leq t_1 < t_2 < \infty$ the increments $W(t_1) - W(t_0)$ and $W(t_2) - W(t_1)$ are independent
3. $\text{Prob}\{W(0) = 0\} = 1$

Wiener Process

- Sample paths of $W(t)$ are continuous functions of t but they do not have bounded variation and are almost everywhere nondifferentiable.
- therefore $\frac{dW(t)}{dt}$ has no meaning in the usual sense
- The Riemann integral $\displaystyle\int_0^t g(\tau)\frac{dW(\tau)}{d\tau}d\tau$ has no meaning since $\frac{dW(\tau)}{d\tau}$ is not defined
- Also $\displaystyle\int_0^t g(\tau)dW(\tau)d\tau$ has no meaning since $W(\tau)$ does not have bounded variation.
- A new definition of a stochastic integral is needed

Itô Stochastic Integral

Assume $f(t)$ is a random function satisfying

$$\int_a^b E(f^2(t))dt < \infty.$$

Let $a = t_1 < t_2 < \cdots < t_k < t_{k+1} = b$ be a partition of $[a, b]$, $\Delta t = t_{i+1} - t_i = (b-a)/k$ and $\Delta W(t_i) = W(t_{i+1}) - W(t_i)$ where $W(t)$ is the standard Wiener process. The Itô stochastic integral of f is

$$\int_a^b f(t)dW(t) = l.i.m._{k\to\infty} \sum_{i=1}^{k} f(t_i)\Delta W(t_i)$$

where $l.i.m$ denotes mean square convergence.

If $F_k = \sum_{i=1}^{k} f(t_i)\Delta W(t_i)$ and $\mathcal{I} = \int_a^b f(t)dW(t)$ then $l.i.m._{k\to\infty} F_k = \mathcal{I}$ means $\lim_{k\to\infty} E[(F_k - \mathcal{I})^2] = 0$

Itô Stochastic Integral

- Mean square convergence

$$E[(F_k - \rangle)^2] \to 0$$

implies convergence in the mean:

$$E(|F_k - \mathcal{I}|) \to 0$$

- The converse is not true

Itô Stochastic Integral

Simple Itô integrals:

- $$\int_a^b dW(t) = W(b) - W(a)$$

- For any well-defined random function $F(W(t), t)$

$$\int_a^b dF(W(t), t) = F(W(b), b) - F(W(a), a)$$

- The Itô stochastic integral is a linear operator on the set of functions f whose Itô stochastic integral exists.

Itô Stochastic Integral

Properties of Itô stochastic integrals

Assume $f(t)$ and $g(t)$ are random functions satisfying $\int_a^b F(f^2(t))dt < \infty$ and α and $a < b < c$ are constants. Then

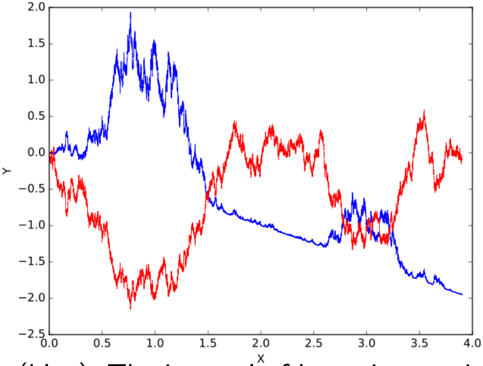
1. $\int_a^b \alpha f(t)dW(t) = \alpha \int_a^b f(t)dW(t)$
2. $\int_a^b \alpha(f(t) + g(t))dW(t) = \int_a^b f(t)dW(t) + \int_a^b g(t)dW(t)$
3. $\int_a^b f(t)dW(t) = \int_a^c f(t)dW(t) + \int_c^b f(t)dW(t)$
4. $E\left[\int_a^b f(t)dW(t)\right] = 0$
5. $E\left[\left(\int_a^b f(t)dW(t)\right)^2\right] = \int_a^b E(f^2(t))dt$

Itô Stochastic Integral

Example

$$\int_0^t W(\tau)dW(\tau) = \frac{1}{2}\left[W^2(t) - t\right]$$

This can be shown using the properties of the Wiener process



Brownian motion (blue), The integral of brownian motion with itself (red).

Itô Stochastic Integral

Example

The following example highlights that the Itô integral is different than the Riemann-Stieltjes integral.

1. Evaluate $\int_a^b W(t)dW(t)$

- Evaluating Itô integrals using the definition can be very difficult
- We will talk about theory allowing us to evaluate some Itô integrals
- Notation:

$$X(t) = \int_0^t W(\tau)dW(\tau)$$

This integral is often written in differential form:

$$dX(t) = W(t)dW(t), \quad X(0) = 0$$

This form is **Itô stochastic differential equation**.

Itô Stochastic Differential Equation

SDE

A stochastic process $\{X(t) : t \in [0, \infty)\}$ is said to satisfy the Itô stochastic differential equations (SDE):

$$dX(t) = \alpha(X(t), t)dt + \beta(X(t), t)dW(t))$$

if for $t \geq 0$ it is a solution of the integral equations:

$$X(t) = X(0) + \int_0^t \alpha(X(\tau), \tau)d\tau + \int_0^t \beta(X(\tau), \tau)dW(\tau)$$

where the first integral is a Riemann integral and the second integral is an Itô stochastic integral.

Itô Stochastic Differential Equation

Example: diffusion with drift

- Consider the diffusion equation with drift. The forward Kolmogorov differential equation is

$$\frac{\partial p}{\partial t} = -c\frac{\partial p}{\partial x} + \frac{D}{2}\frac{\partial^2 p}{\partial x^2}, \qquad x \in (-\infty, \infty)$$

$p(x,0) = \delta(x - x_0)$ with solution

$$p(x,t) = \frac{1}{\sqrt{2\pi Dt}}\exp\left(-\frac{(x - x_0 - ct)^2}{2Dt}\right)$$

- The SDE corresponding to this process is

$$dX(t) = cdt + \sqrt{D}dW(t), \quad X(0) = x_0$$

So that $X(t) \sim N(x_0 + ct, Dt)$.

Itô Stochastic Differential Equation

Example: exponential growth

- Consider the exponential growth model:

$$\frac{dX}{dt} = (\lambda - \mu)X$$

- The SDE representation of the exponential growth model is

$$dX(t) = (\lambda - \mu)X(t)dt + \sqrt{(\lambda + \mu)X(t)}dW(t), \quad X(0) = x_0 > 0$$

- $X(t)$ has a p.d.f that is a solution of the forward Kolmogorov equation

$$\frac{\partial p}{\partial t} = -(\lambda - \mu)\frac{\partial (xp)}{\partial x} + \frac{\lambda + \mu}{2}\frac{\partial^2 (xp)}{\partial x^2}, \quad x \in (0, \infty)$$

$p(x, 0) = \delta(x - x_0)$

- The mean and variance are same for the CTMC model

$$E(X) = X_0 e^{(\lambda - \mu)t} \quad \& \quad Var(X) = X_0 \frac{\lambda + \mu}{\lambda - \mu} e^{(\lambda + \mu)t}(e^{(\lambda + \mu)t} - 1)$$

Itô's Formula is like a "chain rule".

Itô's Formula

Suppose $X(t)$ is a solution of the Itô SDE:

$$dX(t) = \alpha(X(t), t)dt + \beta(X(t), t)dW(t))$$

If $F(x, t)$ is a real-valued function defined for $x \in \mathbb{R}$ and $t \in [a, b]$ with continuous partial derivatives $\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}, \frac{\partial^2 F}{\partial x^2}$ then

$$dF(X(t), t) = f(X(t), t)dt + g(X(t), t)dW(t)$$

where

$$f(x, t) = \frac{\partial F(x, t)}{\partial t} + \alpha(x, t)\frac{\partial F(x, t)}{\partial x} + \frac{1}{2}\beta^2(x, t)\frac{\partial^2 F(x, t)}{\partial x^2}$$

and $g(x, t) = \beta(x, t)\frac{\partial F(x,t)}{\partial x}$

There is a multidimensional Itôs formula for multivariate processes.

Examples: growth with environmental variation

Exponential growth

Consider the SDE

$$dX(t) = rX(t)dt + cX(t)dW(t)$$

1. Apply Itô's formula letting $F(x) = \ln x$
2. Integrate from 0 to t and solve for $X(t)$.

Logistic growth

Consider the SDE

$$dX(t) = rX(t)\left(1 - \frac{X(t)}{K}\right)dt + cX(t)dW(t)$$

1. Give brief explanations of the terms in above model
2. Apply Itô's formula letting $F(x) = \frac{1}{x}$.

Examples using Itô's Formlula

Use Itô's Formlula to verify integrals

1.
$$\int_a^b W(t)dW(t) = \frac{1}{2}[W^2(b) - W^2(a) - (b - a)]$$

2.
$$\int_a^b t dW(t)$$

Deriving Itô SDE from forward Kolmogorov equation

- Under suitable smoothness of the coefficients, a solution of the SDE

$$dX(t) = \alpha(X(t), t)dt + \beta(X(t), t)dW(t)$$

  is a diffusion process.
- That means that it is also a solution of the Kolmogorov equation

$$\frac{\partial p}{\partial t} = -\frac{\partial(\alpha(x,t)p)}{\partial x} + \frac{1}{2}\frac{\partial^2(\beta^2(x,t)p)}{\partial x^2}$$

  where $p(x,t)$ is the p.d.f of the stochastic process.
- Given a forward Kolmogorov equation, in the above form, we can write the SDE.

Question 9 from chp. 8

The forward Kolmogorov equation of a diffusion process has the form

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left[\left((b_1 - d_1)x - (b_2 + d_2)x^2\right) p\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\left((b_1 - d_1)x + (b_2 + d_2)x^2\right) p\right]$$

$x \in (0, \infty), b_i, d_i > 0,$ and $b_1 > d_1$. Write the corresponding SDE.

Deriving Itô SDE from probabilities

- Given the probabilities of events for a stochastic process we can formulate a SDE using the expectation and covariance matrices
- Consider the following probabilities associated with changes in 2 interacting populations

i	$(\Delta X)_i$	Probability, p_i
1	$(1, 0)$	$b_1 \Delta t$
2	$(0, 1)$	$b_2 \Delta t$
3	$(-1, 0)$	$d_1 \Delta t$
4	$(0, -1)$	$d_2 \Delta t$
5	$(-1, 1)$	$m_{21} \Delta t$
6	$(1, -1)$	$m_{12} \Delta t$

Deriving Itô SDE from probabilities

- Step 1: Calculate the expectation

$$E(\Delta X) = \begin{pmatrix} b_1 - d_1 - m_{21} + m_{12} \\ b_2 - d_2 + m_{21} - m_{12} \end{pmatrix} \Delta t$$

- Step 2: Calculate an approximation to the covariance matrix

$$\begin{aligned} \Sigma(\Delta X) &= E([\Delta X][\Delta X^T]) - E(\Delta X)[E(\Delta X)^T] \\ &\approx E \begin{pmatrix} (\Delta X_1)^2 & (\Delta X_1)(\Delta X_2) \\ (\Delta X_1)(\Delta X_2) & (\Delta X_2)^2 \end{pmatrix} \\ &= \begin{pmatrix} b_1 + d_1 + m_{21} + m_{12} & -m_{21} - m_{12} \\ -m_{21} - m_{12} & b_2 + d_2 + m_{21} + m_{12} \end{pmatrix} \Delta t \end{aligned}$$

Deriving Itô SDE from probabilities

- Step 3: Find a matrix B such that $\Sigma = BB^T\Delta t$
  - dimensions of B are (# variables) by (# events)
  - based of the table of probabilities, under the square root
  - Check that $\Sigma = BB^T\Delta t$ after you formulate B

$$B = \begin{pmatrix} \sqrt{b_1} & 0 & -\sqrt{d_1} & 0 & -\sqrt{m_{21}} & \sqrt{m_{12}} \\ 0 & \sqrt{b_2} & 0 & -\sqrt{d_2} & \sqrt{m_{21}} & -\sqrt{m_{12}} \end{pmatrix}$$

- The SDE has the form:

$$dX(t) = \mu(X(t), t)dt + B(X(t), t)dW(t)$$

where μ is the ODE system and $W(t)$ is a vector of independent Wiener process with length of the # events.

Deriving Itô SDE from probabilities

SIR Epidemic Process

Consider the probabilities associated withe changes in the SIR model:

i	$(\Delta X)_i$	Probability, p_i
1	$(-1, 1)$	$\beta SI/N\Delta t$
2	$(0, -1)$	$\gamma I\Delta t$

1. Use the table of parameters to set up a system of SDEs