5 Continuous-Time Markov Chains

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Lecture notes follow: Allen, Linda JS. An introduction to stochastic processes with applications to biology. CRC Press, 2010.

Continuous-Time Markov Chains

- Now we switch from DTMC to study CTMC
- Time in continuous: $t \in [0, \infty)$
- The Random variables are discrete
- Many of biological applications
  - processes with discrete changes that occur at irregular times
  - births and deaths in a population that breeds at different rates throughout the year
  - epidemics
  - gene expression

Definitions and Notation

Let $\{X(t) : t \in [0, \infty)\}$ be a collection of discrete random variables with values in a finite or infinite set, $\{1, 2, ..., N\}$ or $\{0, 1, 2, ...\}$.

- The index set is continuous $t \in [0, \infty)$.

Continuous-Time Markov Chain (CTMC)

The stochastic process $\{X(t) : t \in [0, \infty)\}$ is a CTMC if it satisfies the following conditions for any sequence of real numbers satisfying $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n \leq t_{n+1}$.

$$\begin{aligned}\text{Prob}\{X(t_{n+1}) = i_{n+1} | X(t_0) = i_0, X(t_1) = i_1, ..., X(t_n) = i_n\} \\ = \text{Prob}\{X(t_{n+1}) = i_{n+1} | X(t_n) = i_n\}\end{aligned}$$

- Each random variable $X(t)$ has an associcated probability distribution $\{p_i(t)\}_{i=0}^{\infty}$ where

$$p_i(t) = \text{Prob}\{X(t) = i\}$$

- Let $p(t) = (p_0(t), p_1(t), ...)^T$ be the vector of probabilities

Definitions and Notation

- transition probabilties define the relation between the random variables $X(s)$ and $X(t)$ for $s < t$
- transition probabilites are define below for $i, j = 0, 1, 2, \ldots$:

$$p_{ji}(t,s) = \text{Prob}\{X(t) = j | X(s) = i\}, \qquad s < t$$

- If the transition probabilites don't explicitly depend on s or t but only depend on the length of the time interval $t - s$, they are called **stationary** or **homogeneous**
- otherwise, they are **nonstationary** or **nonhomogeneous**
- We'll assume the transition probabilities are stationary (unless stated otherwise)

Definitions and Notation

Stationary transition probabilities

$$p_{ji}(t-s) = \text{Prob}\{X(t)=j|X(s)=i\}$$
$$= \text{Prob}\{X(t-s)=j|X(0)=i\}$$

for $s < t$. The **transition matrix** is $P(t) = (p_{ji}(t))$ where in most cases, $p_{ji}(t) \geq 0$ and

$$\sum_{j=0}^{\infty} p_{ji}(t) = 1$$

for $t \geq 0$.

$P(t)$ is a stochastic matrix for all $t \geq 0$

Definitions and Notation

The transition probabilities are solutions of the **Chapman-Kolmogorov** equations:

$$\sum_{k=0}^{\infty} p_{jk}(s)p_{ki}(t) = p_{ji}(t+s)$$

or in matrix form:

$$P(s)P(t) = P(s+t)$$

for all $s, t \in [0, \infty)$.

DTMC vs CTMC

- DTMC: There is a *jump* to a new state at discrete times: $1, 2, ...,$
- CTMC: The *jump* can occur at any time $t \geq 0$
  - consider a CTMC beginning at state $X(0)$
  - the process stays in state $X(0)$ for a random amount of time: W_1
  - it then jumps to a new state: $X(W_1)$
  - is stays in state $X(W_1)$ for a random amount of time: W_2
  - it then jumps to a new state $X(W_2)$
- W_i is a random variable for the time of the i^{th} jump.

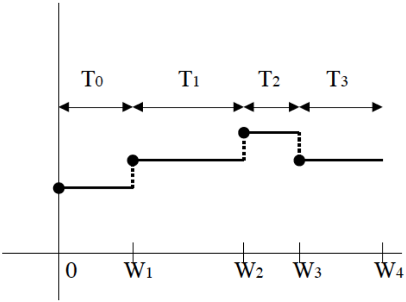
Waiting and Holding Times

Jump times or waiting times

The collection of random variables $\{W_i\}_{i=0}^{\infty}$ defines the jump times or waiting times of the process. Where we define $W_0 = 0$.

Interevent times or holding times or sojourn times

Random variables $T_i = W_{i+1} - W_i$ are the holding times.



Explosive Processes (blow up in finite time)

- Exceptional cases that may occur when the state space is infinite
- The transition matrix $P(t) = (p_{ji}(t))$ property

$$\sum_{j=0}^{\infty} p_{ji}(t) = 1$$

  for $t \geq 0$ does **not** hold.
- The value of the state approaches infinity at a finite time:

$$\lim_{t \to T^-} X(T) = \infty \qquad \text{for } T < \infty$$

- Here $p_{ji}(T) = 0$ for all $i, j = 0, 1, 2, ...$ which means that

$$\sum_{j=0}^{\infty} p_{ji}(T) = 0$$

- These cases are *exceptional*. All Finite CTMCs are nonexplosive and most well-known birth and death processes are nonexplosive.

Transition Matrix

- An important difference between the treatment of discrete-time and continuous-time Markov chains is that in the latter case there is no one canonical transition matrix that is used to characterize the entire process.
- Instead, we can define an entire family of transition matrices indexed by time.
- Here $p_{ji}(t) = p(t, j, i) = \text{Prob}\{X(t) = j | X(0) = i\}$ are elements of matrix $P(t)$

$$P(t) = \begin{pmatrix} p_{00}(t) & p_{01}(t) & p_{02}(t) & \cdots \\ p_{10}(t) & p_{11}(t) & p_{12}(t) & \cdots \\ p_{20}(t) & p_{21}(t) & p_{22}(t) & \cdots \\ p_{30}(t) & p_{31}(t) & p_{32}(t) & \cdots \\ \vdots & \vdots & \vdots & \ldots \end{pmatrix}$$

Poisson Process

Poisson Process

A CTMC $\{X(t) : t \in [0, \infty)\}$ with state space $\{0, 1, 2, ...\}$ and the following properties

1. For $t = 0$, $X(0) = 0$
2. For Δt sufficiently small, the transition probabilites are:

$$\begin{aligned} p_{i+1,i}(\Delta t) &= \text{Prob}\{X(t+\Delta t) = i+1 | X(t) = i\} = \lambda \Delta t + o(\Delta t) \\ p_{ii}(\Delta t) &= \text{Prob}\{X(t+\Delta t) = i | X(t) = i\} = 1 - \lambda \Delta t + o(\Delta t) \\ p_{ji}(\Delta t) &= \text{Prob}\{X(t+\Delta t) = j | X(t) = i\} = o(\Delta t), \qquad j \geq i+2 \\ p_{ji}(\Delta t) &= 0, \qquad j < i \end{aligned}$$

Functions $p_{i+1,i}(\Delta t) - \lambda \Delta t$, $p_{ii}(\Delta t) - 1 + \lambda \Delta t$, and $p_{ji}(\Delta t)$ are $o(\Delta t)$ as $\Delta t \to 0$ (*"little oh of Δt"*). These are known as **infinitesimal transition probabilities**

Poisson Process

Infinitesimal transition probabilities:

$$\lim_{\Delta t \to 0} = \frac{p_{i+1,i}(\Delta t) - \lambda \Delta t}{\Delta t} = 0$$

$$\lim_{\Delta t \to 0} = \frac{p_{ii}(\Delta t) - 1 + \lambda \Delta t}{\Delta t} = 0$$

$$\lim_{\Delta t \to 0} = \frac{p_{ji}(\Delta t)}{\Delta t} = 0, \qquad j \geq i + 2$$

Poisson Process: transition matrix

$$P(\Delta t) = \begin{pmatrix} p_{00}(\Delta t) & p_{01}(\Delta t) & p_{02}(\Delta t) & \cdots \\ p_{10}(\Delta t) & p_{11}(\Delta t) & p_{12}(\Delta t) & \cdots \\ p_{20}(\Delta t) & p_{21}(\Delta t) & p_{22}(\Delta t) & \cdots \\ p_{30}(\Delta t) & p_{31}(\Delta t) & p_{32}(\Delta t) & \cdots \\ \vdots & \vdots & \vdots & \ldots \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda\Delta t & 0 & 0 & \cdots \\ \lambda\Delta t & 1-\lambda\Delta t & 0 & \cdots \\ 0 & \lambda\Delta t & 1-\lambda\Delta t & \cdots \\ 0 & 0 & \lambda\Delta t & \cdots \\ \vdots & \vdots & \vdots & \ldots \end{pmatrix} + o(\Delta t)$$

- Note the columns sum to 1.
- For small Δt the process can either stay in the same state or move to the next larger state $i \to i+1$.
- The probability that the process moves up 2 or more states is small and approaches 0 when $\Delta t \to 0$.

Poisson Process: derived system of DEs

Derive a system of differential equations for $p_i(t)$ for $i = 0, 1, 2, ...$
Since $X(0) = 0$ it follows that

$$p_{i0}(t-0) = \text{Prob}\{X(t) = i | X(0) = 0\} = \text{Prob}\{X(t) = i\} = p_i(t)$$

Thus $p_{i0}(t) = p_i(t)$. It follows that

$$p_0(t + \Delta t) = p_0(t)[1 - \lambda \Delta t + o(\Delta t)]$$

Subtracting $p_0(t)$, dividing by Δt, and taking the limit as $\Delta t \to 0$ yields:

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t)$$

The initial conditions comes from: $p_0(0) = \text{Prob}\{X(0) = 0\} = 1$.
Solving this system yields:

$$p_0(t) = e^{-\lambda t}$$

Poisson Process: derived system of DEs

Similarly,

$$p_i(t+\Delta t) = p_i(t)[1-\lambda\Delta t + o(\Delta t)] + p_{i-1}(t)[\lambda \Delta t + o(\Delta t)] + o(\Delta t)$$

leads to

$$\frac{dp_i(t)}{dt} = -\lambda p_i(t) + \lambda p_{i-1}(t), \qquad p_i(0) = 0, \qquad i \geq 1$$

a system of differential-difference equations.
The system can be solved sequentially beginning with $p_0(t) = e^{-\lambda t}$ to show that

$$p_1(t) = \lambda t e^{-\lambda t}$$

$$p_2(t) = (\lambda t)^2 \frac{e^{-\lambda t}}{2!}$$

$$\vdots$$

$$p_i(t) = (\lambda t)^i \frac{e^{-\lambda t}}{i!}$$

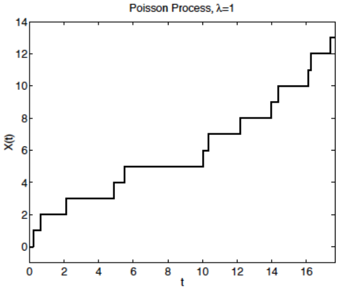
Poisson Process

The probability distribution $\{p_i(t)\}_{i=1}^{\infty}$ gives a Poisson distribution with parameter λt

$$p_i(t) = (\lambda t)^i \frac{e^{-\lambda t}}{i!}, \qquad i = 0, 1, 2, ...$$

with mean and variance:

$$m(t) = \sigma^2(t) = \lambda t$$



Poisson Process: holding time

- The probability $p_0(t) = e^{-\lambda t}$ can be thought of as a waiting time probability
- Its the probability that the first event $0 \to 1$ occurs at a time greater than t.
- Let W_1 be the random variable for the time until the process reaches state 1 (the holding time until the first jump)

$$\text{Prob}\{W_1 > t\} = e^{-\lambda t} \qquad \text{or} \qquad \text{Prob}\{W_1 \leq t\} = 1 - e^{-\lambda t}$$

- W_1 is an exponential random variable with parameter λ
- In general, it can be shown that the holding time has an exponential distribution.
- We will see that this is true in general for Markov processes.

Generator Matrix Q

Basic Ideas

- transition probabilities p_{ji} are used to derived transition rates q_{ji}
- transition rates form the infinitesimal generator matrix Q
- matrix Q defines a relationship between the rates of change of the transition probabilites

Generator Matrix Q

Derivations

- assume transition probabilities p_{ji} are continuous and differentiable for $t \geq 0$
- assume at $t = 0$ the following holds:

$$p_{ji}(0) = 0, \qquad j \neq i \qquad \text{and} \qquad p_{ii}(0) = 1$$

- for $j \neq i$, define

$$q_{ji} = \lim_{\Delta t \to 0^+} \frac{p_{ji}(\Delta t) - p_{ji}(0)}{\Delta t} = \lim_{\Delta t \to 0^+} \frac{p_{ji}(\Delta t)}{\Delta t}$$

- and define

$$q_{ii} = \lim_{\Delta t \to 0^+} \frac{p_{ii}(\Delta t) - p_{ii}(0)}{\Delta t} = \lim_{\Delta t \to 0^+} \frac{p_{ii}(\Delta t) - 1}{\Delta t}$$

Generator Matrix Q

Derivations

- since $\sum_{j=0}^{\infty} p_{ji} = 1$ it follows that

$$1 - p_{ii}(\Delta t) = \sum_{j=0, j\neq i}^{\infty} p_{ji}(\Delta t) = \sum_{j=0, j\neq i}^{\infty} [q_{ji}\Delta t + o(\Delta t)]$$

- Then

$$q_{ii} = \lim_{\Delta t \to 0^+} \frac{p_{ii}(\Delta t) - 1}{\Delta t} = \lim_{\Delta t \to 0^+} \frac{-\sum_{j=0, j\neq i}^{\infty} [q_{ji}\Delta t + o(\Delta t)]}{\Delta t} = -\sum_{j=0, j\neq i}^{\infty} q_{ji}$$

- Note that is can be shown that $\sum_{j\neq i} o(\Delta t) = o(\Delta t)$

Generator Matrix Q

Derivations

In general we can express the relationship between p_{ji} and q_{ji} as

$$p_{ji}(\Delta t) = \delta_{ji} + q_{ji}(\Delta t) + o(\Delta t)$$

where δ_{ji} is Kronecker's delta symbol.*

Or in matrix form:

$$Q = \lim_{\Delta t \to 0^+} \frac{P(\Delta t) - I}{\Delta t}$$

where $P(\Delta t) = (p_{ji}(\Delta t)$ is the infinitesimal transition matrix and I is the identity matrix (with appropriate dimensions).

$^\star\delta_{j]} = 1$ for $j = i$ and $\delta_{j]} = 0$ for $j \neq i$.

Generator Matrix Q

Quick Summary

$$p_{ji}(\Delta t) = \delta_{ji} + q_{ji}(\Delta t) + o(\Delta t)$$

$$Q = \lim_{\Delta t \to 0^+} \frac{P(\Delta t) - I}{\Delta t}$$

- The probability that the process moves from its current state i to another state j during a short period of time Δt is approximately proportional to the amount of time elapsed.
- In other words, q_{ji} is the rate at which transitions occur from state i to state j

$$q_{ii} = -\sum_{j=0, j\neq i}^{\infty} q_{ji}$$

Generator Matrix Q

Definition

infinitesimal generator matrix

The matrix of transition rates $Q = q_{ji}$ is

$$Q = \begin{pmatrix} q_{00} & q_{01} & q_{02} & \cdots \\ q_{10} & q_{11} & q_{12} & \cdots \\ q_{20} & q_{21} & q_{22} & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

$$= \begin{pmatrix} -\sum_{i=1}^{\infty} q_{i0} & q_{01} & q_{02} & \cdots \\ q_{10} & -\sum_{i=1}^{\infty} q_{i1} & q_{12} & \cdots \\ q_{20} & q_{21} & -\sum_{i=1}^{\infty} q_{i2} & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

Generator Matrix Q

Properties

$$Q = \begin{pmatrix} -\sum_{i=1}^{\infty} q_{i0} & q_{01} & q_{02} & \cdots \\ q_{10} & -\sum_{i=1}^{\infty} q_{i1} & q_{12} & \cdots \\ q_{20} & q_{21} & -\sum_{i=1}^{\infty} q_{i2} & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

Properties:

- The columns sum to 0
- The diagonal elements are the negative sum of the off-diagonal elements in the column.

Generator matrix for Poisson process

The infinitesimal transition matrix for the Poisson process

$$P(\Delta t) = \begin{pmatrix} 1-\lambda\Delta t & 0 & 0 & \cdots \\ \lambda\Delta t & 1-\lambda\Delta t & 0 & \cdots \\ 0 & \lambda\Delta t & 1-\lambda\Delta t & \cdots \\ 0 & 0 & \lambda\Delta t & \cdots \\ \vdots & \vdots & \vdots & \dots \end{pmatrix} + o(\Delta t)$$

Taking the limit $Q = \lim_{\Delta t \to 0^+} \frac{P(\Delta t) - I}{\Delta t}$ yields

$$Q = \begin{pmatrix} -\lambda & 0 & 0 & \cdots \\ \lambda & -\lambda & 0 & \cdots \\ 0 & \lambda & -\lambda & \cdots \\ 0 & 0 & \lambda & \cdots \\ \vdots & \vdots & \vdots & \dots \end{pmatrix}$$

Generator matrix for Poisson process

The differential-difference equations for the Poisson process we saw earlier:

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t)$$

$$\frac{dp_i(t)}{dt} = -\lambda p_i(t) + \lambda p_{i-1}(t), \qquad i \geq 1$$

can be expressed in terms of the generator matrix Q:

$$\frac{dp(t)}{dt} = Qp(t)$$

Embedded Markov Chain

Recall:

- Sample paths of CTMC spend a random amount of time in each state before jumping to a new state
- W_i for $i = 0, 1, 2, ...$ are the waiting times
- $T_i = W_{i+1} - W_i$ are the holding times

Embedded Markov Chain

Let Y_n denote the random variable for the state of the CTMC $\{X(t) : t \in [0, \infty)\}$ at the nth jump,

$$Y_n = X(W_n), \qquad n = 0, 1, 2, ...$$

The set of random variables $\{Y_n\}_{n=0}^{\infty}$ is the **embedded markov chain** or the jump chain associated with the CTMC $\{X(t) : t \in [0, \infty)\}$.

Embedded Markov Chain

- The embedded Markov chain is a DTMC
- It is useful for classifying the states of the corresponding CTMC
- Define a transition matrix $T = (t_{ji})$ for the embedded Markov chain:

$$t_{ji} = \text{Prob}\{Y_{n+1} = j | Y_n = i\}$$

Embedded MC for Poisson Process

Consider the Poisson process, where $X(0) = X(W_0) = 0$ and $X(W_n) = n$ for $n = 1, 2, ...$ The embedded Markov chain $\{Y_n\}$ satisfies $Y_n = n, n = 0, 1, 2, ...$ The transition from state n to $n + 1$ occurs with probability 1. The transition matrix $\{Y_n\}$ is

$$T = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

Embedded Markov Chain

- Matrix T can be defined using the generator matrix Q:

$$t_{ii} = \begin{cases} 0 & \text{if } q_{ii} \neq 0 \\ 1 & \text{if } q_{ii} = 0 \end{cases}$$

$$t_{ji} = \begin{cases} \frac{-q_{ji}}{q_{ii}} & \text{if } q_{ii} \neq 0 \\ 0 & \text{if } q_{ii} = 0 \end{cases} \qquad \text{for } j \neq i$$

- where state i is absorbing if $q_{ii} = 0$ (rate of change is 0).

Embedded Markov Chain

Transition Matrix of the Embedded Markov Chain

For $q_{ii} \neq 0$ for $i = 0, 1, 2, \ldots$

$$T = \begin{pmatrix} 0 & -\frac{q_{01}}{q_{11}} & -\frac{q_{02}}{q_{22}} & \cdots \\ -\frac{q_{10}}{q_{00}} & 0 & -\frac{q_{12}}{q_{22}} & \cdots \\ -\frac{q_{20}}{q_{00}} & -\frac{q_{21}}{q_{11}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

If any $q_{ii} = 0$ then $t_{ii} = 1$ and the other elements in that column are 0.

- Matrix T is stochastic
- Transition probabilities are independent of n (homogeneous)
- $T^n = \left(t_{ji}^{(n)}\right)$ where $t_{ji}^{n} = \text{Prob}\{Y_n = j | Y_0 = i\}$

Example

Suppose a finite CTMC has a generator matrix given by

$$Q = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

1. What is the Transition matrix of the corresponding embedded Markov Chain?
2. Is the embedded Markov Chain aperiodic or periodic?
3. Is the corresponding CTMC periodic?

Classifications of States

- Classifications for states of CTMC are similar to those of DTMC
- Transition probabilities $P(t) = (P_{ji}(t))$ and the transition matrix for the embedded MC $T = (t_{ji})$ are used to define the classification schemes

Basic Definitions

- State j can be **reached** from state i, $i \to j$, if $p_{ji}(t) > 0$ for some $t \geq 0$.
- State i **communicates** with state j, $i \leftrightarrow j$, if $i \to j$ and $j \to i$.
- The set of states that communicate is called a **communication class**.
- If every state can be reached from every other state, the Markov chain is **irreducible** otherwise, it is said to be **reducible**
- A set of states C is **closed** if it is impossible to reach any state outside of C from a state inside C, $p_{ji}(t) = 0$ for $t \geq 0$ if $i \in C$ and $j \notin C$.

Classifications of States

- If $p_{ji}(\Delta t) = \delta_{ji} + q_{ji}\Delta t + o(\Delta t)$, then $p_{ji}(\Delta t) > 0$ iff $q_{ji} > 0$ for $j \neq i$ and Δt sufficiently small.
- Therefore, $i \leftrightarrow j$ in the CTMC iff $i \leftrightarrow j$ in the embedded Markov chain.
- The generator matrix Q in the CTMC is irreducible (reducible) iff the transition matrix T in the embedded Markov chain is irreducible (reducible).

Classifications of States

First Return Time

Let T_{ii} be the first time the chain is in state i after leaving state i

$$T_{ii} = \inf\{t > W_1, X(t) = i | X(0) = i\}$$

Here T_{ii} is a continuous random variable called the **first return time**. It can occur at any time $t > 0$.

Recurrent and Transient States

State i is recurrent (transient) in a CTMC $\{X(t)\}, t \geq 0$, if the first return time is finite (infinite),

$$\text{Prob}\{T_{ii} < \infty | X(0) = i\} = 1 (< 1).$$

Classifications of States

- The recurrent and transient definitions in a CTMCs are similar to those in DTMCs
- Recall in the DTMC, state i is said to be recurrent (transient) in a DTMC $\{Y_n\}$, with $Y_0 = i$, if

$$\sum_{n=0}^{\infty} f_{ii}^{(n)} = 1 (< 1)$$

where $f_{ii}^{(n)}$ is the probability that the first return to state i is at step n.

Classifications of States

This theorem relates recurrent and transient states in CTMCs to recurrent and transient states in the corresponding embedded Markov chains.

Theorem 5.1

State i in a CTMC $\{X(t)\}$, $t \geq 0$, is recurrent (transient) iff state i in the corresponding embedded Markov chain $\{Yn\}$, $n = 0, 1, 2, ...$, is recurrent (transient).

- Recurrence or transience in a CTMC can be determined from the properties of the embedded DTMC and its transition matrix T.

Classifications of States

Theorem 5.1, Corollary 1

A state i in a CTMC $\{X(t)\}$, $t \geq 0$, is recurrent (transient) iff

$$\sum_{n=0}^{\infty} t_{ii}^{(n)} = \infty (< \infty)$$

where $t_{ii}^{(n)}$ is the (i, i) element in the transition matrix of T^n of the embedded Markov chain $\{Yn\}$.

Theorem 5.1, Corollary 2

In a finite CTMC, all states cannot be transient and in addition, if the finite CTMC is irreducible, the chain is recurrent.

Classifications of States

Poisson Process Example

The transition matrix of the embedded MC for the Poisson process is:

$$T = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

- Here $\lim_{n\to\infty} T^n = \mathbf{0}$ (lower triangular).
- For sufficiently large n and all i, $t_{ii}^{(n)} = 0$, which implies $\sum_{n=0}^{\infty} t_{ii}^{(n)} < \infty$
- Therefore, every state is transient in the Poisson process.
- This is an obvious result since each state $X(W_i) = i$ can only advance to state $i+1$, $X(W_{i+1}) = i+1$
- a return to state i is impossible.

Classifications of States

- Unfortunately, the concepts of null recurrence and positive recurrence for a CTMC cannot be defined in terms of the embedded Markov chain
- Positive recurrence depends on the waiting times $\{W_i\}$
- The embedded Markov chain alone is not sufficient to define positive recurrence.

Positive and Null Recurrence

State i is **positive recurrent (null recurrent)** in the CTMC $\{X(t) : t \in [0, \infty)\}$ if the mean recurrence time is finite (infinite):

$$\mu_{ii} = E(T_{ii}|X(0) = i) < \infty (= \infty).$$

- This definition is not actually very useful
- The basic limit theorems for DTMCs and CTMCs are more useful ways to determine μ_{ii} in order to classify states as positive or null recurrent.

Classifications of States

Recall the basic limits theorems for DTMCs

Basic Limit Theorem, Aperiodic DTMC

Let $\{Y_n\}_{n=0}^{\infty}$ be a recurrent, irreducible, and aperiodic DTMC with transition matrix $T = (t_{ij})$:

$$\lim_{n\to\infty} t_{ij}^{(n)} = \frac{1}{\mu_{ii}}$$

Basic Limit Theorem, Periodic DTMC

Let $\{Y_n\}_{n=0}^{\infty}$ be a recurrent, irreducible, and d-periodic DTMC, $d > 1$, with transition matrix $T = (t_{ij})$:

$$\lim_{n\to\infty} t_{ii}^{(nd)} = \frac{d}{\mu_{ii}}$$

- Note, there is no concept of aperiodic and periodic in CTMC because the holding time is random.

Basic Limit Theorem for CTMCs

If the CTMC $\{X(t) : t \in [0, \infty)\}$ is nonexplosive and irreducible, then for all i and j,

$$\lim_{t\to\infty} p_{ji}(t) = -\frac{1}{q_{ii}\mu_{ii}}$$

where $0 < \mu_{ii} \leq \infty$ is the mean recurrence time. In particular, a finite and irreducible CTMC is nonexplosive and the above limit exists and is positive.

Corollary

A finite, irreducible CTMC is positive recurrent.

- The result differs from DTMC due to the term q_{ii} in the limit.
- This term is needed to define the limit. μ_{ii} has units of time and q_{ii} has units $1/\text{time}$.

Kolmogorov Differential Equations

- The forward and backward Kolmogorov DE show the rate of change on the transition probabilites
- For the forward DE $p_{ji}(t + \Delta t)$ is expanded using the Chapman-Kolmogorov equations

$$p_{ji}(t + \Delta t) = \sum_{k=0}^{\infty} p_{jk}(\Delta t) p_{ki}(t)$$

- Since $p_{ji}(\Delta t) = \delta_{ji} + q_{ji}\Delta t + o(\Delta t)$, we rewrite this as

$$p_{ji}(t + \Delta t) = \sum_{k=0}^{\infty} p_{ki}(t)[\delta_{jk} + q_{jk}\Delta t + o(\Delta t)]$$

- subtract $p_{ji}(t)$ from both sides, divide by Δt, and take limit $\Delta t \to \infty$

$$\frac{p_{ji}(t)}{dt} = \sum_{k=0}^{\infty} q_{ji} p_{ki}(t), \qquad i, j = 0, 1, \ldots$$

Kolmogorov Differential Equations

Forward Kolmogorov Diffferential Equation

$$\frac{dP(t)}{dt} = QP(t)$$

where $P(t) = (P_{ji}(t))$ is the matrix of transition probabilities and $Q = (q_{ji})$ is the generator matrix.

- In physics and chemistry, this is referred to as the **master equations**
- In the case that the initial distribution of the process satisfies $X(0) = k$ ($p_i(0) = \delta_{ik}$), then the transition probability $p_{ik}(t)$ is the same as the state probability $p_i(t) = \text{Prob}\{X(t) = i | X(0) = k\}$. In this case,

  $$\frac{dp(t)}{dt} = Qp(t)$$

  where $p(t) = (p_0(t), p_1(t), ...)^T$.

Kolmogorov Differential Equations

- The system of equations

$$\frac{dp(t)}{dt} = Qp(t)$$

can be approximated by a system of difference equations

$$p(n+1) = Pp(n)$$

which are the forward equations corresponding to the DTMC.
- This shows the relationship between the Kolmogrov differential equations and DTMC models.

Kolmogorov Differential Equations

- Derivation of the backward Kolmogorov DE is similar
- For the backward DE $p_{ji}(t + \Delta t)$ is expanded using the Chapman-Kolmogorov equations

$$p_{ji}(t + \Delta t) = \sum_{k=0}^{\infty} p_{ki}(\Delta t) p_{jk}(t)$$

- Since $p_{ji}(\Delta t) = \delta_{ji} + q_{ji}\Delta t + o(\Delta t)$, we rewrite this as

$$p_{ji}(t + \Delta t) = \sum_{k=0}^{\infty} p_{jk}(t)[\delta_{ki} + q_{ki}\Delta t + o(\Delta t)]$$

- subtract $p_{ji}(t)$ from both sides, divide by Δt, and take limit $\Delta t \to \infty$

$$\frac{p_{ji}(t)}{dt} = \sum_{k=0}^{\infty} p_{jk}(t) q_{ki}, \qquad i, j = 0, 1, ...$$

Kolmogorov Differential Equations

Backward Kolmogorov Diffferential Equation

$$\frac{dP(t)}{dt} = P(t)Q$$

where $P(t) = (P_{ji}(t))$ is the matrix of transition probabilities and $Q = (q_{ji})$ is the generator matrix.

- The backward Kolmogorov DEs are useful in first passage time problems
  - distributions for the time it takes to reach a specific state
  - reaching a specific state for the first time
  - These types of problems depend on the initial state of the process
  - These are similar to problems we did for DTMC (mean first passage time, expected duration)

Kolmogorov Differential Equations

- These differential equations depend on the existence of the generator matrix Q.
- For finite Markov chains, Q always exists.
- The solution P(t) can be found via the forward or backward equations.
- In birth and death chains and other applications, the transition matrix $P(t)$ is defined such that the forward and backward Kolmogorov differential equations can be derived.

Stationary Probability Distribution

- The Kolmogorov differential Equations

$$\frac{dP(t)}{dt} = QP(t) \qquad \& \qquad \frac{dP(t)}{dt} = P(t)Q$$

  can be used to define a stationary probability distribution π

- π can be defined in terms of the generator matrix Q or the transition matrix $P(t)$

Stationary Probability Distribution

π in terms of Q

Let $\{X(t) : t \in [0, \infty)\}$ be a CTMC with generator matrix Q. Suppose $\pi = (\pi_0, \pi_1, ..., \pi)^T$ is nonnegative and

$$Q\pi = 0 \qquad \& \qquad \sum_{i=0}^{\infty} \pi_i = 1$$

Then π is called the stationary probability distribution of the CTMC.

π in terms of $P(t)$

Let $\{X(t) : t \in [0, \infty)\}$ be a CTMC with transition matrix $P(t)$.

$$P(t)\pi = \pi, \qquad t \geq 0 \qquad \& \qquad \sum_{i=0}^{\infty} \pi_i = 1, \qquad \pi_i \geq 0$$

for $i = 0, 1, 2, ...$

Stationary Probability Distribution

- These 2 deffinitions of π (in terms of Q and $P(t)$) are equivalent only if the transition matrix $P(t)$ is a solution of the forward and backward Kolmogorov equations.

$$\frac{dP(t)}{dt} = QP(t) \qquad \& \qquad \frac{dP(t)}{dt} = P(t)Q$$

- This is always the case for finite CTMC
- If the CTMC is nonexplosive, positive recurrent, and irreducible then π is the limiting distribution in the Basic Limit theorem
- The Basic limit theorem for aperiodic DTMC can be extended to CTMC

Stationary Probability Distribution

Theorem 5.3

Let $\{X(t) : t \in [0, \infty)\}$ be a nonexplosive, possitive recurrent, and irreducible CTMC with transition matrix $P(t) = (p_{ji}(t))$ and generator matrix $Q = (q_{ji})$, then there exiss a unique postive stationary probability distribution π where $Q\pi = 0$ such that

$$\lim_{t\to\infty} p_{ij}(t) = \pi_i, \qquad i, j = 1, 2, ...$$

It follows that the mean recurrence time can be computed from the stationary distribution:

$$\pi_i = -\frac{1}{q_{ii}\mu_{ii}}$$

Stationary Probability Distribution

Example

$$Q = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

1. Determine the unique positive stationary probability distribution.
2. What are the mean recurrence times?

Finite Markov Chains

Corollary 5.2

Let $\{X(t) : t \in [0, \infty)\}$ be a finite and irreducible CTMC with transition matrix $P(t) = (p_{ji}(t))$ and generator matrix $Q = (q_{ji})$, then there exiss a unique postive stationary probability distribution π where $Q\pi = 0$ such that

$$\lim_{t \to \infty} p_{ij}(t) = \pi_i = -\frac{1}{q_{ii}\mu_{ii}}, \qquad i, j = 1, 2, ..., N$$

- Note: this is similar to the previous theorem but the assumptions of nonexplosiveness and recurrence are not needed with the CTMC is finite.

Finite Markov Chains

Suppose the generator matrix of a CTMC with 2 states is

$$Q = \begin{pmatrix} -a & b \\ a & -b \end{pmatrix}$$

where $a, b > 0$.

1. Is the CTMC reducible or irreducible?
2. What is the unique stationary probability distribution?
3. What are the mean recurrence times?

Finite Markov Chains

- Sometimes it is possible to find an explicit solution to the forward and backward Kolmogorov equations.
- Assume the state space of a finite Markov chain is $\{0, 1, 2, ..., N\}$ and the infinitesimal transition probabilities satisfy

$$p_{ji}(\Delta t) = \delta_{ji} + q_{ji}(t) + o(\Delta t)$$

- The Kolmogorov differential Equations

$$\frac{dP(t)}{dt} = QP(t) \qquad \& \qquad \frac{dP(t)}{dt} = P(t)Q$$

  with $P(0) = I$ have the unique solution:

$$P(t) = e^{Qt}P(0) = e^{Qt}$$

- Here e^{Qt} is the matrix exponential:

$$e^{Qt} = I + Qt + q^2\frac{t^2}{2!} + Q^3\frac{t^3}{3!} + \cdots = \sum_{k=0}^{\infty} Q^k\frac{t^k}{K!}$$

Methods to calculate the matrix exponential e^{Qt}

Method 1

Suppose Q is an $n \times n$ diagonalizable matrix with eigenvalues $\lambda_i, i = 1, 2, ..., n$. Then $Q^k = H\Lambda^k H^{-1}$ where $\Lambda = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$ and the columns of H are the right eigenvectors of Q:

$$P(t) = e^{Qt} = H \sum_{k=0}^{\infty} \Lambda^k \frac{t^k}{k!} H^{-1} = H\text{diag}\left(e^{\lambda_1 t}, e^{\lambda_2 t}, ..., e^{\lambda_n t}\right) H^{-1}$$

Method 2 to calculate the matrix exponential e^{Qt}

Suppose Q is an $n \times n$ with characteristic equation

$$det(\lambda I - Q) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 = 0$$

which is also a characteristic equation of the differential equation

$$x^{(n)}(t) = a_{n-1}x^{(n-1)}(t) + \cdots + a_0 x(t) = 0$$

To find e^{Qt}, find n linearly independent solutions $x_1(t), x_2(t), ..., x_n(t)$ with the initial conditions

$$\begin{cases} x_1(0) = 1 \\ x_1'(0) = 0 \\ \vdots \\ x_1^{(n-1)}(0) = 0 \end{cases}, \begin{cases} x_2(0) = 0 \\ x_2'(0) = 1 \\ \vdots \\ x_2^{(n-1)}(0) = 0 \end{cases}, \ldots, \begin{cases} x_n(0) = 0 \\ x_n'(0) = 0 \\ \vdots \\ x_n^{(n-1)}(0) = 1 \end{cases}$$

Then $P(t) = e^{Qt} = x_1(t)I + x_2(t)Q + \cdots + x_n(t)Q^{n-1}$

Example

Suppose the generator matrix of a CTMC with 2 states is

$$Q = \begin{pmatrix} -a & b \\ a & -b \end{pmatrix}$$

where $a, b > 0$.

1. Use the first method to calculate $P(t) = e^{Qt}$
2. Use the second method to calculate $P(t) = e^{Qt}$
3. What is $\lim_{t\to\infty} P(t)$?
4. How does this matrix relate to the stationary probability distribution?

Example

Suppose the generator matrix of a CTMC is

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

1. What is the transition matrix T of the embedded DTMC?
2. Is Q irreducible or reducible?
3. Given the expression for e^{Qt}, what is $\lim_{t\to\infty} e^{Qt}$?

$$e^{Qt} = \begin{pmatrix} 1 & 1-e^{-t} & 1-2e^{-t}+e^{-2t} & 1-3e^{-t}+3e^{-2t}-e^{-3t} \\ 0 & e^{-t} & 2e^{-t}-2e^{-2t} & 3e^{-t}-6e^{-2t}+3e^{-3t} \\ 0 & 0 & e^{-2t} & 3e^{-2t}-3e^{-3t} \\ 0 & 0 & 0 & e^{-3t} \end{pmatrix}$$

4. How does this matrix relate to the stationary probability distribution?

Generating Function Technique

- Heres is another method for getting information about the probability distribution associates with a CTMC
- A PDE is derived so the solutions of the equation is a generating function
- Depending on the equation, the solution is either a
  - probability generating function (p.g.f)
  - moment generating function (m.g.f)
  - cumuland generating function (c.g.f)

Generating Function Technique

probability generating function (p.g.f)

$$\mathcal{P}(z,t) = \sum_{i=0}^{\infty} p_i(t) z^i$$

moment generating function (m.g.f)

$$M(\theta,t) = \sum_{i=0}^{\infty} p_i(t) e^{\theta i}$$

cumulant generating function (c.g.f)

$$K(\theta,t) = \ln M(\theta,t)$$

The generating functions depend on 2 continuous variables z and t or θ and t.

Generating Function Technique

Mean

The mean $m(t)$ of the process at time t is

$$m(t) = \left.\frac{\partial \mathcal{P}(z,t)}{\partial z}\right|_{z=1} = \sum_{i=0}^{\infty} i p_i(t)$$

or in terms of the m.g.f and c.g.f. :

$$m(t) = \left.\frac{\partial M(\theta,t)}{\partial \theta}\right|_{\theta=0} = \left.\frac{\partial K(\theta,t)}{\partial \theta}\right|_{\theta=0}$$

Generating Function Technique

Variance

The variance $\sigma^2(t)$ at time t is

$$\sigma^2(t) = \left.\frac{\partial^2 \mathcal{P}(z,t)}{\partial z^2}\right|_{z=1} + \left.\frac{\partial \mathcal{P}(z,t)}{\partial z}\right|_{z=1} - \left(\left.\frac{\partial \mathcal{P}(z,t)}{\partial z}\right|_{z=1}\right)^2$$

or in terms of the m.g.f and c.g.f. :

$$\sigma^2(t) = \left.\frac{\partial^2 M(\theta,t)}{\partial \theta^2}\right|_{\theta=0} - \left(\left.\frac{\partial M(\theta,t)}{\partial \theta}\right|_{\theta=0}\right)^2 = \left.\frac{\partial^2 K(\theta,t)}{\partial \theta^2}\right|_{\theta=0}$$

Generating Function Technique

- A partial differential equation (PDE) is derived from the forward Kolmogorov equations.
- The p.g.f is a solution of this PDE
- When the initial distribution is a fixed value, then the forward Kolmogorov equations can be expressed in terms of the state probabilities

$$\frac{dp}{dt} = Qp$$

$$\frac{dp_i(t)}{dt} = \sum_{k=0}^{\infty} q_{ik} p_k(t), \qquad i = 0, 1, 2, \dots$$

Generating Function Technique

Deriving the PDEs

Starting with the forward Kolmogorov equations

$$\frac{dp_i(t)}{dt} = \sum_{k=0}^{\infty} q_{ik} p_k(t), \qquad i = 0, 1, 2, \ldots$$

we can derive the following PDE using the p.g.f.

$$\frac{\partial \mathcal{P}(z,t)}{\partial t} = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} q_{ik} p_k(t) z^i$$

Here we multiplied the equation by z^i, summed over i, and interchanged the order of differentiation and summations.

Generating Function Technique

Deriving the PDEs

- A PDE for the m.g.f can be derived with the same technique, but multiply with $e^{i\theta}$ instead of z^i
- Alternatively, the PDE for the m.g.f can be derived directly from the PDE for the p.g.f with a change a variables
  - Recall that $M(\theta, t) = \mathcal{P}(e^{\theta}, t)$
  - Therefor $z = e^{\theta}$
- A PDE for the c.m.f can be derived from the PDE of the m.g.f by letting $K(\theta, t) = \ln M(\theta, t)$
- The generating function technique is often used with birth and death chains
- If the PDEs are first-order, they can be solved by the method of characteristics.

Poisson process Example

The forward Kolmogorov differential equations for the Poisson process are

$$\frac{dp_i(t)}{dt} = -\lambda p_i(t) + \lambda p_{i-1}(t), \qquad i \geq 1$$
$$\frac{dp_0(t)}{dt} = -\lambda p_0(t)$$

1. Derive the PDE for the p.g.f.
2. Use the known initial conditions $\mathcal{P}(z, 0) = 1$ to solve the PDE and get an expression for the p.g.f.
3. Make a change of variables to formulate the m.g.f
4. Take the ln to formulate the c.g.f.

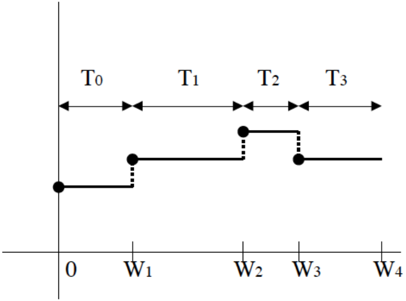
Note that these generating functions correspond to the Poisson distribution with parameter λt.

Interevent Time and Stochastic Realizations

- In order to calculate sample paths of a CTMC we need to know the distribution of for the time between successive events or the **interevent time**

Recall: Interevent times or holding times or sojourn times

Random variables $T_i = W_{i+1} - W_i \geq 0$ are the interevent times. Here W_i is the time of the ith jump.



Interevent Time

- We will now show for CTMC the Interevent Time has an Exponential Distribution
- First lets look at the theorem (Theorem 5.4).
- Then we will look at the steps of the proof.

Interevent Time Theorem

Let $\{X(t) : t \in [0, \infty)\}$ be a CTMC such that

$$\sum_{j \neq n} p_{jn}(\Delta t) = \alpha(n)\Delta t + o(\Delta t)$$

and

$$p_{nn}(\Delta t) = 1 - \alpha(n)\Delta t + o(\Delta t)$$

for Δt sufficiently small. Then the interevent time $T_i = W_{i+1} - W_i$, given $X(W_i) = n$, is an exponential random variable with parameter $\alpha(n)$. The c.d.f. for T_i is

$$F_i = 1 - e^{-\alpha(n)t}$$

so that the mean and variance of T_i are

$$E(T_i) = \frac{1}{\alpha(n)} \qquad \& \qquad \text{Var}(T_i) = \frac{1}{[\alpha(n)]^2}$$

Interevent Time Theorem Proof

- Assume $X(W_i) = n$ (The process is at state n at time the ith jump).
- Let $\alpha(n)\Delta t + o(\Delta t)$ be the probability that the process moves to a different state in Δt:

$$\sum_{j \neq n} p_{jn}(\Delta t) = \alpha(n)\Delta t + o(\Delta t)$$

- The probability of no change in state is then $1 - \alpha(n)\Delta t + o(\Delta t)$:

$$p_{nn}(\Delta t) = 1 - \alpha(n)\Delta t + o(\Delta t)$$

- Let $G_i(t)$ be the probability that the process remains in state n for a time length of t, that is $[W_i, W_{i+t}]$:

$$G_i(t) = \text{Prob}\{t + W_i < W_{i+1}\}$$

- $G_i(t)$ can be written in terms of the interevent time:

$$G_i(t) = \text{Prob}\{T_i > t\}$$

Interevent Time Theorem Proof

- For Δt sufficiently small

$$G_i(t+\Delta t) = G_i(t)p_{nn}(\Delta t) = G_i(t)(1 - \alpha(n)\Delta t + o(\Delta t))$$

- subtracting $G_i(t)$ from both sides and taking the limit as $\Delta t \to 0$:

$$\frac{dG_i(t)}{dt} = -\alpha(n)G_i(t)$$

- The initial condition for this ODE is $G_i(0) = \text{Prob}\{T_i > 0\} = 1$.
- This is a first-order homogeneous differential equation has solution:

$$G_i(t) = \text{Prob}\{T_i > t\} = e^{-\alpha(n)t}$$

- Thus the probability that $T_i \leq t$ for $t \geq 0$ is:

$$\text{Prob}\{T_i \leq t\} = 1 - G_i(t) = 1 - e^{-\alpha(n)t} = F_i(t)$$

Interevent Time Theorem Proof

$$F_i(t) = 1 - e^{-\alpha(n)t}$$

- Function $F_i(t)$ is the cumulant distribution function for the interevent time T_i
- it corresponds to an exponential random variable with parameter $\alpha(n)$.
- The p.g.f. for T_i is $F_i'(t) = f_i(t) = \alpha(n)e^{-\alpha(n)t}$.
- Recall that the mean and variance for an exponential random variable with parameter $\alpha(n)$ are

$$E(T_i) = \frac{1}{\alpha(n)} \qquad \& \qquad \text{Var}(T_i) = \frac{1}{[\alpha(n)]^2}$$

Interevent Time Theorem

Let $\{X(t) : t \in [0, \infty)\}$ be a CTMC such that

$$\sum_{j \neq n} p_{jn}(\Delta t) = \alpha(n)\Delta t + o(\Delta t)$$

and

$$p_{nn}(\Delta t) = 1 - \alpha(n)\Delta t + o(\Delta t)$$

for Δt sufficiently small. Then the interevent time $T_i = W_{i+1} - W_i$, given $X(W_i) = n$, is an exponential random variable with parameter $\alpha(n)$. The c.d.f. for T_i is

$$F_i = 1 - e^{-\alpha(n)t}$$

so that the mean and variance of T_i are

$$E(T_i) = \frac{1}{\alpha(n)} \qquad \& \qquad \text{Var}(T_i) = \frac{1}{[\alpha(n)]^2}$$

Interevent Time Theorem Examples

Examples of birth and death processes

Consider a birth process with birth probability $b_n \Delta t + o(\Delta t)$ and $X(W_i) = n$.

1. What is the mean waiting time until another birth occurs?

Consider a birth and death process with birth probability $b_n \Delta t + o(\Delta t)$ and death probability $d_n \Delta t + o(\Delta t)$ and $X(W_i) = n$.

1. What is the mean waiting time until another event (a birth or a death) occurs?
2. When an event occurs, what is the probability that the event will be a birth?
3. When an event occurs, what is the probability that the event will be a death?

Stochastic Realizations

- The random variable T_i can be expressed in terms of the distribution function $F_i(t)$ and a uniform random variable U.
- This will be useful for simulating sample paths

Theorem 5.5

Let U be a uniform random variable defined on $[0,1]$ and T be a continuous random variable defined on $[0,\infty)$ with $\text{Prob}\{T \leq t\} = F(t)$. Then $T = F^{-1}(U)$, where F is the cumulative distribution of the random variable T.

For $F(t) = 1 - e^{-\alpha(n)t}$

$$T = F^{-1}(U) = -\frac{\ln(1-U)}{\alpha(n)} = -\frac{\ln(U)}{\alpha(n)}$$

Simple Birth and Death Process

- $X(t)$ is a random variable for the total population size at time t
- Two events can occur
  - birth event: $i \to i+1$
  - death event: $i \to i-1$
- For Δt sufficiently small the transition probabilities are

$$p_{i+j,i}(\Delta t) = \text{Prob}\{\Delta x(t) = j | X(t) = i\}$$
$$= \begin{cases} di\Delta t + o(\Delta t), & j = -1 \\ bi\Delta t + o(\Delta t), & j = 1 \\ 1 - (b+d)i\Delta t + o(\Delta t), & j = 0 \\ o(\Delta t), & j \neq -1, 0, 1 \end{cases}$$

Given $X(W_i) = n, \alpha(n) = (b+d)n$.

1. What is the interevent time T_i?
2. What is the probability that the next event will be a birth?

Simple Birth and Death Process

- The deterministic analogue of this simple birth and death process is the differential equation:

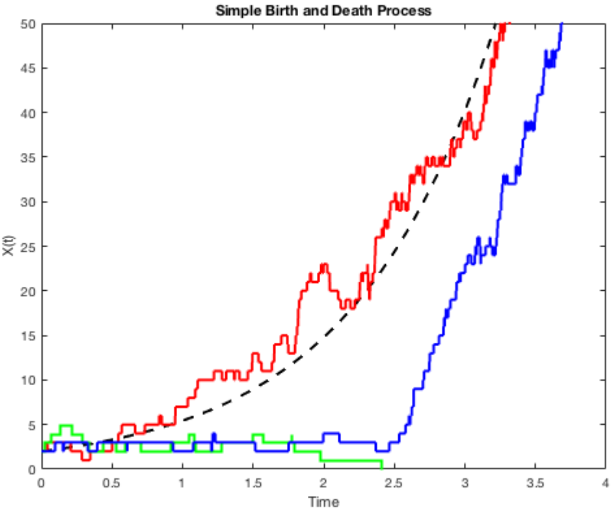
$$\frac{dn}{dt} = (b - d)n, \qquad n(0) = N$$

- This has solution:

$$n(t) = Ne^{b-d}t$$

- We can compare stochastic realizations, or sample paths, with this deterministic solution

Simple Birth and Death Process



Gillespie Algorithm

Simulating Sample Paths

- It was created by Joseph L. Doob and others (circa 1945)
- and popularized by Dan Gillespie in 1976, 1977 where he uses it to simulate chemical or biochemical systems of reactions
- The advantage of this approach is that rather than generating multiple exponential random variables, one for each possible transition out of the current state, we only need to generate two random variables, one for the holding time and one for the next state.

Gillespie Algorithm

Simulating Sample Paths

Gillespie Algorithm

1. **Initialization**: Initialize the number of molecules in the system, reaction constants, and random number generators.
2. **Monte Carlo step**: Generate random numbers to determine the next reaction to occur as well as the time interval. The probability of a given reaction to be chosen is proportional to the number of substrate molecules
3. **Update**: Increase the time step by the randomly generated time in Step 2. Update the molecule count based on the reaction that occurred.
4. **Iterate**: Go back to Step 2 unless the number of reactants is zero or the simulation time has been exceeded.

Gillespie Algorithm

Simulating Sample Paths

- Gillespie developed the **direct method**
- Examples of this are in the appendix of Chp. 5
- Here 2 uniform random variables are needed per iteration
  - One to simulate the time to the next event
  - The other to choose the event
- This method works well when population sizes are small, but becomes costly for large population sizes
- Many modifications and adaptations exist: next reaction method (Gibson & Bruck), tau-leaping, as well as hybrid techniques