2 Discrete-Time Markov Chains

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Biomathematics II
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Lecture notes follow: Allen, Linda JS. An introduction to stochastic processes with applications to biology. CRC Press, 2010.

Definitions and Notation

Discrete-time stochastic process $\{X_n\}_{n=0}^{\infty}$

- X_n is discrete random variables on finite or countably infinite state space
- n index is used for time $\{0, 1, 2, ...\}$

Discrete-Time Markov Chain (DTMC)

$\{X_n\}_{n=0}^{\infty}$ has the Markov property if

$$\text{Prob}\{X_n = i_n | X_0 = i_0, .., X_{n-1} = i_{n-1}\} = \text{Prob}\{X_n = i_n | X_{n-1} = i_{n-1}\}$$

and the process is called a DTMC.

- Notation: $\text{Prob}\{\cdot\} = P_{X_n}\{\cdot\}$ used. (P will represent a transition matrix).
- $\{p_i(n)\}_{n=0}^{\infty}$ is the p.m.f. associate with X_n, where $p_i(n) = \text{Prob}\{X_n = i\}$.
- Transition probabilities relate state of process at time n to $n+1$ (X_n to X_{n+1}).

Definitions and Notation

One-step Transition Probability

$$p_{ji}(n) = \text{Prob}\{X_{n+1} = j | X_n = i\}$$

is the probability that the process is in state j at time $n+1$ given that the process was in state i at time n.
For each state, p_{ji} satisfies

$$\sum_{j=1}^{\infty} p_{ji} = 1 \qquad \& \qquad p_{ji} \geq 0.$$

- The above summation means the process at state i must transfer to j or stay in i during the next time interval.
- p_{ji} don't depend on time they are *stationary* or *homogeneous*
- $p_{ji}(n)$ do depend on time they are *nonstationary* or *nonhomogeneous*

Definitions and Notation

Transition Matrix

The DTMC $\{X_n\}_{n=0}^{\infty}$ with one-step transition probabilities $\{p_{ij}\}_{i,j=1}^{\infty}$ has transition matrix $P = (p_{ij})$:

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1j} & \cdots \\ p_{21} & p_{22} & \cdots & p_{2j} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ p_{i1} & p_{i2} & \cdots & p_{ij} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

- Columns sum to 1, since $\sum_{j=1}^{\infty} p_{ji} = 1$.
- Called a *Stochastic Matrix*.
- Note notation: p_{ij} is the probability of transition from state j to state i (other sources may define this differently).

Definitions and Notation

N-step Transition Probability

$$p_{ji}^{(n)} = \text{Prob}\{X_n = j | X_0 = i\}$$

is the probability of transferring from state i to state j in n time steps. The n-step transition matrix $P^{(n)} = \left(p_{ji}^{(n)}\right)$, where $p_{ji}^{(1)} = p_{ji}$ and

$$p_{ji}^{(0)} = \delta_{ji} = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases}$$

where δ_{ji} is the *Kronecker delta symbol*. Then $P^{(1)} = P$ and $P^{(0)} = I$ the identity matrix.

Definitions and Notation

Chapman-Kolmogorov Equations

$$p_{ji}^{(n)} = \sum_{k=1}^{\infty} p_{jk}^{(n-s)} p_{ki}^{(s)}, \qquad 0 < s < n$$

Or in terms of matrix notation:

$$P^{(n)} = P^{(n-s)} P^{(s)}$$

Here,

$$\begin{aligned} P^{(1)} &= P \\ P^{(2)} &= P^{(1)} P^{(1)} = P^2 \\ &\vdots \\ P^{(n)} &= P^n \end{aligned}$$

The n-step transition matrix $P^{(n)}$ is just the nth power of P.

Definitions and Notation

Let $p(n) = (p_1(n), p_2(n), \ldots)^T$ be the vector form of the p.m.f. for X_n where $p_i(n) = \text{Prob}\{X_n = i\}$. The probabilities satisfy

$$\sum_{i=1}^{\infty} p_i(n) = 1.$$

The probability distribution associated with X_{n+1} can be found:

$$p_i(n+1) = \sum_{j=1}^{\infty} p_{ij} p_j(n) \quad \text{or} \quad p(n+1) = Pp(n)$$

This projects the process *forward* in time.

$$p(n+m) = P^{n+m} p(0) = P^n (P^m p(0)) = P^n p(m)$$

Example

Etterson et al. 2009* propose a simple Markov chain model to describe the reproductive activities of a single female bird in a single breeding season. A female can occupy one of four states:

1. actively nesting
2. successfully fledged a brood
3. failed to fledge a brood
4. completed all nesting activities for the season

The state space is $E = \{1, 2, 3, 4\}$ and the random variable $X_n \in E$ represents the state of the female following the nth change of state.

*Etterson, Matthew A., et al. "Markov chain estimation of avian seasonal fecundity." Ecological Applications 19.3 (2009): 622-630.

Example

Etterson et al. 2009 used the transition matrix:

$$P = \begin{pmatrix} 0 & s^a & 1-s^a & 0 \\ 1-q_s & 0 & 0 & q_s \\ 1-q_f & 0 & 0 & q_f \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0.369 & 0.631 & 0 \\ 0.33 & 0 & 0 & 0.67 \\ 0.58 & 0 & 0 & 0.42 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where s is the daily nest survival probability, a is the average time from first egg to fledging, q_s is the probability that a female quits breeding following a successful breeding attempt, and q_f is the probability that a female quits breeding following a failed breeding attempt. The numbers are estimates obtained from field studies of a population of Eastern Meadowlarks in Illinois.

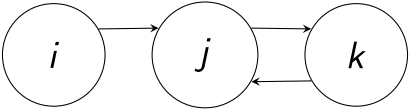
Classification of States

Communicate

State j can be reached from state i is there is a nonzero probability $p_{ji}^{(n)} > 0$ for some $n \geq 0$. This is denoted as $i \to j$. If $i \to j$ and $j \to i$, i and j are said to communicate, or be in the same class, denoted as $i \leftrightarrow j$; that is, there exists n and n' such that

$$p_{ji}^{(n)} 0 \quad \& \quad p_{ij}^{(n')} > 0$$

Directed Graph:



Here, $i \to j$ as $p_{ji} > 0$ and $i \to k$ as $p_{ki}^{(2)} > 0$ but is is not that case that $k \to i$.

Classification of States

Equivalence Relation

$i \leftrightarrow j$ is an equivalence relation on the state $\{1, 2, ...\}$

1. reflexivity: $i \leftrightarrow i$ (because $p_{ii}^{(0)} = 1$)
2. symmetry: $i \leftrightarrow j$ implies $j \leftrightarrow i$
3. transitivity: $i \leftrightarrow j$, $j \leftrightarrow k$ implies $i \leftrightarrow k$.

Communication Classes

The set of equivalences classes in a DTMC are the communication classes. If every state in the Markov chain can be reached by every other state, then there is only one communication class.

Irreducible

If there is only one communication class, then the Markov chain is irreducible, otherwise is it reducible.

Classification of States

Irreducible

A transition matrix P is irreducible if the directed graph is strongly connected. It is reducible if the directed graph is not strongly connected.

Closed

Set of states C is closed if it is impossible to reach any state outside of C from any state in C by one-step transitions: $p_{ji} = 0$ if $i \in C$ and $j \notin C$.

If C is a closed communicating class for a Markov chain X, then that means that once X enters C, it never leaves C.

Absorbing State

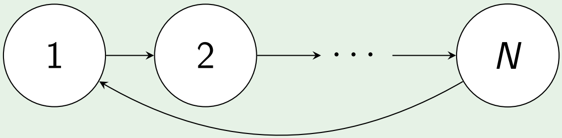
State i is absorbing if $p_{ii} = 1$.

If i is an absorbing state once the process enters state i, it is trapped there forever.

Classification of States

Example

$$P = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$



The Markov chain is irreducible and it periodic with period N (beginning in state i, it takes N steps to return to state i: $P^N = I$)

Classification of States

Periodic

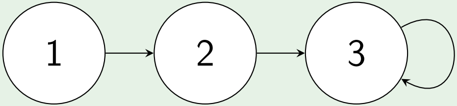
The period of state i is the greatest common divisor of all $n \geq 1$ for which $p_{ii}^{(n)} > 0$:

$$d(i) = g.c.d.\{n | p_{ii}^{(n)} > 0 \text{ and } n \geq 1\}.$$

If $d(i) > 1$ the state is periodic of period $d(i)$. If $d(i) = 1$ the state is aperiodic. If $p_{ii}^{(n)} = 0$ for all $n \geq 1$ define $d(i) = 0$.

Example

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

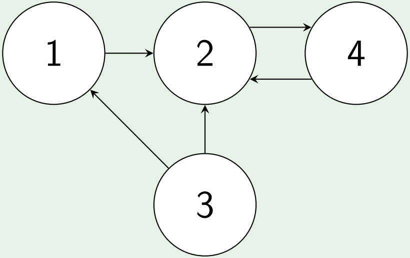


Communication classes: $\{1\}, \{2\}, \{3\}$. $d(i) = 0$ for $i = 1, 2$ because here $p_{ii}^{(n)} = 0$. State 3 is aperiodic since $d(3) = 1$.

Classification of States

Example

$$P = \begin{pmatrix} 0 & 0 & p_{13} & 0 \\ p_{21} & 0 & p_{23} & p_{24} \\ 0 & 0 & 0 & 0 \\ 0 & p_{42} & 0 & 0 \end{pmatrix}$$



3 communication classes: $\{1\}, \{3\}, \{2, 4\}$. Markov chain is reducible.

Classification of States

Etterson et al. 2009

Etterson et al. 2009 used the transition matrix:

$$P = \begin{pmatrix} 0 & s^a & 1-s^a & 0 \\ 1-q_s & 0 & 0 & q_s \\ 1-q_f & 0 & 0 & q_f \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $s \in (0,1), a > 0$, and $q_s, q_f \in (0,1)$. State space $E = \{1,2,3,4\}$ includes two communicating classes: $C_1 = \{1,2,3\}$ and $C_2 = \{4\}$. The Markov chain is reducible. C_2 is closed and 4 is an absorbing state.

First Return

First Return Probability

Let $f_{ii}^{(n)}$ be the probability that starting from state i, $X_0 = i$, the first return to state i is at the nth time step:

$$f_{ii}^{(n)} = \text{Prob}\{X_n = i, X_m \neq i, m = 1, 2, ..., n-1 | X_0 = i\}, \quad n \geq 1.$$

The probabilities $f_{ii}^{(n)}$ are the **first return probabilities.** $f_{ii}^{(0)} = 0$.

$f_{ii}^{(1)} = p_{ii}$, but generally $f_{ii}^{(n)} \neq p_{ii}^{n}$.

First Return

Recurrent State

State i is recurrent if

$$\sum_{n=1}^{\infty} f_{ii}^{(n)} = 1.$$

Let random variable T_{ii} be the first return time. Then $\{f_{ii}^{(n)}\}_{n=0}^{\infty}$ defines a probability distributions for T_{ii}. Here, $T_{ii} = n$ with probability $f_{ii}^{(n)}$.

Transient State

State i is transient if

$$\sum_{n=1}^{\infty} f_{ii}^{(n)} < 1.$$

Then $\{f_{ii}^{(n)}\}_{n=0}^{\infty}$ is not a complete set of probabilities needed to define a probability distribution. Here, let $f_{ii} = \sum_{n=0}^{\infty} f_{ii}^{(n)} < 1$ and define $1 - f_{ii}$ as the probability of never returning to i. T_{ii} is the *waiting time* until the chain returns to i.

First Return

Mean Recurrence Time

For recurrent state i, the mean recurrence time is the mean of the distribution T_{ii}:

$$\mu_{ii} = E(T_{ii}) = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$$

For the recurrent state i if $\mu_{ii} < \infty$ it is positive recurrent. If $\mu_{ii} = \infty$, it is null recurrent.

- Ex: absorbing state i. Here, $p_{ii} = 1$ thus $f_{ii}^{(1)} = 1$ and $f_{ii}^{(n)} = 0$ for $n \neq 1$. Therefore $\mu_{ii} = 1$.
- The mean recurrence time for a transient state is infinity: $T_{ii} = \infty$ with probability $1 - f_{ii}$

First Return

Example

Consider a two state Markov chain with transition matrix:

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

where $0 < p_{ii} < 1$ for $i = 1, 2$. Show that both states are positive recurrent.

First Passage Time

First Passage Time Probability

Let $f_{ji}^{(n)}$ be the probability that starting from state i, $X_0 = i$, the first return to state j is at the nth time step:

$$f_{ji}^{(n)} = \text{Prob}\{X_n = j, X_m \neq j, m = 1, 2, ..., n-1 | X_0 = i\}, \quad j \neq i, n \geq 1.$$

The probabilities $f_{ji}^{(n)}$ are the **first passage time probabilities.** $f_{ji}^{(0)} = 0$.

First Passage Form State j from State i

If $\sum_{n=0}^{\infty} f_{ji}^{(n)} = 1$ then $\{f_{ji}^{(n)}\}$ defines a probability distribution for a random variable T_{ji}, the first passage to state j from state i.

Mean First Passage Time

If $X_0 = i$, the mean first passage time to state j is:
$\mu_{ji} = E(T_{ji}) = \sum_{n=1}^{\infty} n f_{ji}^{(n)}$, $j \neq i$

Relationships between step transition and first return probabilities:

$$p_{ii}^{(n)} = \sum_{k=1}^{n} f_{ii}^{(k)} p_{ii}^{(n-k)} \qquad \& \qquad p_{ji}^{(n)} = \sum_{k=1}^{n} f_{ji}^{(k)} p_{ji}^{(n-k)}$$

Generating Functions

Let the generating function for the sequence $\{f_{ji}^{(n)}\}$ be

$$F_{ji}(s) = \sum_{n=0}^{\infty} f_{ji}^{(n)} s^n, \quad |s| < 1$$

Let the generating function for the sequence $\{p_{ji}^{(n)}\}$ be

$$P_{ji}(s) = \sum_{n=0}^{\infty} p_{ji}^{(n)} s^n, \quad |s| < 1$$

Relationships: $F_{ii}(s)P_{ii}(s) = P_{ii}(s) - 1 \qquad \& \qquad F_{ji}(s)P_{jj}(s) = P_{ji}(s)$

Basic Theorems for Markov Chains

Theorem 2.2

A state i is recurrent (transient) if and only if $\sum_{n=0}^{\infty} p_{ii}^{(n)}$ diverges (converges), i.e.

$$\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty (< \infty)$$

The Proof of this used the following theorem:

Abel's Convergence Theorem

$$\text{If } \sum_{k=0}^{\infty} a_k \text{ converges, then } \lim_{s \to 1^-} \sum_{k=0}^{\infty} a_k s^k = \sum_{k=0}^{\infty} a_k = a$$

$$\text{If } a_k \geq 0 \text{ and } \lim_{s \to 1^-} \sum_{k=0}^{\infty} a_k s^k = a \leq \infty, \text{ then } \sum_{k=0}^{\infty} a_k = a$$

Basic Theorems for Markov Chains

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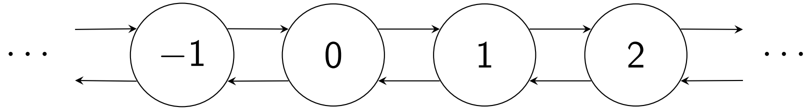
Corollaries

- Assume $i \leftrightarrow j$. State i is recurrent (transient) if and only if state j is recurrent (transient)
- Every recurrent class in a DTMC is a closed set

Infinite Random Walk



$p > 0$ is the probability of moving to the right: $p_{i+1,i} = p$
$q > 0$ is the probability of moving to the left: $p_{i,i+1} = q$
$p + q = 1$

Example Questions

1. Is the MC reducible or irreducible?
2. Is the MC aperiodic or periodic?
3. Assume $p = q = \frac{1}{2}$, is the MC transient or recurrent?
4. Assume $p \neq q$, is the MC transient or recurrent?

Basic Theorems for Markov Chains

Basic Limit Theorem, Aperiodic Markov Chains

Let $\{X_n\}_{n=0}^{\infty}$ be a recurrent, irreducible, and aperiodic DTMC with transition matrix $P = (p_{ij})$:

$$\lim_{n\to\infty} p_{ij}^{(n)} = \frac{1}{\mu_{ii}}$$

Basic Limit Theorem, Periodic Markov Chains

Let $\{X_n\}_{n=0}^{\infty}$ be a recurrent, irreducible, and d-periodic DTMC, $d > 1$, with transition matrix $P = (p_{ij})$:

$$\lim_{n\to\infty} p_{ii}^{(nd)} = \frac{d}{\mu_{ii}}$$

Summary of Classification Schemes

Markov chains or classes can be classified as

Periodic or Aperiodic

Then further classified as

Transient or Recurrent

Then recurrent MC can be classified as

Null recurrent or Positive recurrent.

"*Equilibirum*" of the Markov Chain

Stationary Probability Distribution

A stationary probability distribution is the vector $\pi = (\pi_1, \pi_2, ...)^T$ with:

$$P\pi = \pi \qquad \& \qquad \sum_{i=1}^{\infty} \pi_i = 1.$$

For a finite MC π is an eigenvector of P with eigenvalue $\lambda = 1$:

$$P\pi = \lambda\pi \qquad \& \qquad \sum_{i=1}^{N} \pi_i = 1$$

If a chain is initially at a stationary probability distribution $p(0) = \pi$, then $p(n) = P^n\pi = \pi$ for all time n.

"*Equilibirum*" of the Markov Chain

There may be more than one linearly independent eigenvector for $\lambda = 1$. In this case, the stationary probability distribution is not unique. However a positive recurrent, irreducible, and aperiodic DTMC has a unique stationary probability distribution:

Theorem 2.5

Let $\{X_n\}_{n=0}^{\infty}$ be a positive recurrent, irreducible, and aperiodic DTMC. There is a unique positive stationary probability distribution π with $P\pi = \pi$

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_i, \qquad i, j = 1, 2, \ldots$$

The basic limit theorem then yields:

$$\pi_i = \frac{1}{\mu_{ii}} > 0$$

where μ_{ii} is the mean recurrence time for state i.

"*Equilibirum*" of the Markov Chain

Example

$$P = \begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{pmatrix}$$

1. What is the stationary probability distribution for P?
2. What are the mean recurrence times?

Finite Markov Chains

In finite DTMC, there are **NO null recurrent states** and **not all states can be transient**.

- 4 classification schemes: periodic or aperiodic and transient or positive recurrent
- An irreducible finite DTMC is positive recurrent

In a finite DTMC, a class is recurrent if and only if it is closed

Grey vs. Red Squirrel

Red Squirrel are native to areas of Great Britain and Gray Squirrels invaded many regions in the 19^{th} century. Each region is classified as being in one the following states:

1. occupied by Red squirrels only
2. occupied by Gray squirrels only
3. occupied by both
4. no squirrels

The transitions between states over a period of 1 year were estimated for the following transition matrix:

$$P = \begin{pmatrix} 0.8797 & 0.0382 & 0.0527 & 0.0008 \\ 0.0212 & 0.8002 & 0.0041 & 0.0143 \\ 0.0981 & 0.0273 & 0.8802 & 0.0527 \\ 0.0010 & 0.1343 & 0.0630 & 0.9322 \end{pmatrix}$$

Grey vs. Red Squirrel

$$P = \begin{pmatrix} 0.8797 & 0.0382 & 0.0527 & 0.0008 \\ 0.0212 & 0.8002 & 0.0041 & 0.0143 \\ 0.0981 & 0.0273 & 0.8802 & 0.0527 \\ 0.0010 & 0.1343 & 0.0630 & 0.9322 \end{pmatrix}$$

The eigenvector corresponding to the eigenvalue $\lambda = 1$:

$$\pi = (0.1705, 0.0560, 0.3421, 0.4314)^T$$

1. Describe the squirrel population in the regions over the long run.
2. Determine and interpret the mean recurrence times

Mean First Passage Time

Method to calculate mean first passage time and time until absorption:

$$M = (\mu_{ij}) = \begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1N} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_{N1} & \mu_{N2} & \cdots & \mu_{NN} \end{pmatrix}$$

M is the matrix of mean first passage times. The time it takes to go from $i \to j$ is described below:

$$\mu_{ji} = p_{ji} + \sum_{k=1, k \neq j}^{N} p_{ki}(1 + \mu_{jk}) = 1 + \sum_{k=1, k \neq j}^{N} p_{ki}\mu_{jk}$$

where j is reached in 1 time step with probability p_{ji} or it takes multiple time steps and goes through state k. In Matrix form (E is matrix of 1s):

$$M = E + (M - \mathrm{diag}(M))P$$

Can solve this system (N^2 equations and N^2 unknowns)

Mean First Passage Time

Suppose the MC has k absorbing states. Partition the matrix into k absorbing states and $m - k$ transient states:

$$P = \begin{pmatrix} I & A \\ 0 & T \end{pmatrix}$$

Lemma 2.2

Submatrix $T = (t_{jk})$ of transition matrix P, where indices j, k are from the set of transient states has the following property

$$\lim_{n \to \infty} T^n = \mathbf{0}.$$

Mean First Passage Time

Let v_{ij} be the random variable for the # of visits (before absorption) to the transient state i beginning from j. The expected # of visits to i from j is

$$(E[v_{ij}]) = I + T + T^2 + T^3 + \cdots = (I - T)^{-1}$$

Fundamental Matrix

This is the **Fundamental Matrix** in DTMC:

$$F = (I - T)^{-1}$$

The expected time to absorption is the time spent in each of the transient states. Therefore, we can calculate the time to absorption by summing the columns of F:

Expected Time Untill Absorptions

$$m = \mathbf{1}^T F$$

where $\mathbf{1}$ is a column vector of ones.

Example

Consider the MC with transition matrix:

$$P = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

1. What are the communication classes? Classify each class?
2. Reorder the states and determine matrix T.
3. What is the fundamental matrix?
4. What is the mean time until absorption?

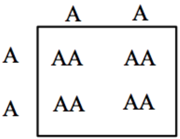
Genetics Inbreeding Problem

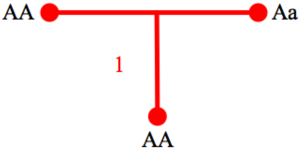
- An allele is a variant form of a gene
- Suppose there are 2 types of alleles for a given gene: a and A
- A diploid individual (2 sets of chromosomes) can have 3 different genotypes or combinations of alleles: AA, aa, Aa
- Assume 2 individuals are randomly mated. Then the next generation of their offsprings (brother and sister) are randomly mated. This inbreeding process continues each year.

Genetics Inbreeding Problem

- Let the mating types be states of a DTMC
- There are 6 states:

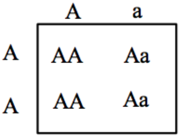
1. $AA \times AA$
2. $AA \times Aa$
3. $Aa \times Aa$
4. $Aa \times aa$
5. $AA \times aa$
6. $aa \times aa$

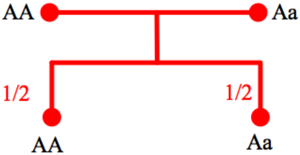




- Suppose parents are type 1: $AA \times AA$
- Next generation of offsprings are all AA
- Next generation mating combinations are all type 1
- $p_{11} = 1$

Genetics Inbreeding Problem



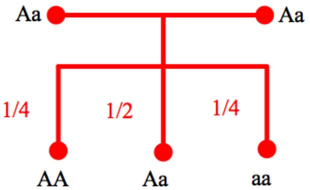


- Suppose parents are type 2: $AA \times Aa$
- Next generation of offsprings are $1/2$ AA and $1/2$ Aa
- Next generation mating combinations are $(AA \times AA)$, $(AA \times Aa)$, $(Aa \times Aa)$
  - proportion of matings of type $(AA \times AA) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$
  - proportion of matings of type $(AA \times Aa)^{\star} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2}$
  - proportion of matings of type $(Aa \times Aa) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$
- $p_{12} = 1/4, p_{22} = 1/2, p_{32} = 1/4$

$^{\star}(AA \times Aa)$ and $(Aa \times AA)$

Genetics Inbreeding Problem

	A	a
A	AA	Aa
a	Aa	aa



- Suppose parents are type 3: $Aa \times Aa$
- Next generation of offsprings are $1/4$ AA, $1/2$ Aa, and $1/4$ aa
- Next generation mating combinations are $(AA \times AA)$, $(AA \times Aa)$, $(Aa \times Aa)$, $(Aa \times aa)$, $(AA \times aa)$, and $(aa \times aa)$.
  - proportion of matings of type $(AA \times AA) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{16}$
  - proportion of matings of type $(AA \times Aa) = \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{4}$
  - proportion of matings of type $(Aa \times Aa) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$
  - proportion of matings of type $(Aa \times aa) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$
  - proportion of matings of type $(AA \times aa) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{8}$
  - proportion of matings of type $(aa \times aa) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{16}$
- $p_{13} = 1/16$, $p_{23} = 1/4$, $p_{33} = 1/4$, $p_{43} = 1/4$, $p_{53} = 1/8$, $p_{63} = 1/16$

Genetics Inbreeding Problem

$$p = \begin{pmatrix} 1 & 1/4 & 1/16 & 0 & 0 & 0 \\ 0 & 1/2 & 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1 & 0 \\ 0 & 0 & 1/4 & 1/2 & 0 & 0 \\ 0 & 0 & 1/8 & 0 & 0 & 0 \\ 0 & 0 & 1/16 & 1/4 & 0 & 1 \end{pmatrix}$$

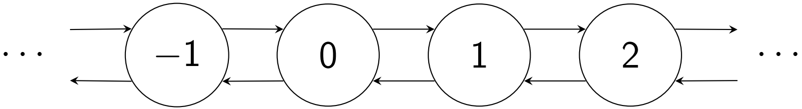
$$p = \begin{pmatrix} 1 & | & 1/4 & 1/16 & 0 & 0 & | & 0 \\ - & - & - & - & - & - & - & - \\ 0 & | & 1/2 & 1/4 & 0 & 0 & | & 0 \\ 0 & | & 1/4 & 1/4 & 1/4 & 1 & | & 0 \\ 0 & | & 0 & 1/4 & 1/2 & 0 & | & 0 \\ 0 & | & 0 & 1/8 & 0 & 0 & | & 0 \\ - & - & - & - & - & - & - & - \\ 0 & | & 0 & 1/16 & 1/4 & 0 & | & 1 \end{pmatrix} = \begin{pmatrix} 1 & A & 0 \\ \mathbf{0} & T & \mathbf{0} \\ 0 & B & 0 \end{pmatrix}$$

Questions

1. What are the communication classes? Classify them.
2. Are there any absorbing states?
3. Determine the expected time until absorption.
4. What are the probabilities of absorption into states 1 and 6?

Unrestricted Random Walk in Higer Dimensions

Recall infinite random walk in 1D:



$p > 0$ is the probability of moving to the right: $p_{i+1,i} = p$
$q > 0$ is the probability of moving to the left: $p_{i,i+1} = q$
$p + q = 1$

Properties

1. MC is irreducible
2. MC is periodic with period $=2$
3. Assume $p = q = \frac{1}{2}$, then the MC is recurrent
4. Assume $p \neq q$, then the MC is transient

Unrestricted Random Walk in Higer Dimensions

- 1 Dimension:
  - Chain is null recurrent if and only if $p = q = 1/2$
  - Probability of moving left equals probability of moving right
- 2 Dimensions:
  - If probabilities of moving in any direction are equal ($1/4$ for up, down, right, and left) then the chain is null recurrent
- 3 Dimensions:
  - If probabilities of moving in any direction are equal ($1/6$ for up, down, right, left, forward, and backward) then the chain is **transient**
  - A path along a line or in a plane is much more restricted than in space
  - Behavior is 3 or higher dimensions in more complicated