Hopf algebras

Let F be a field, and let H be an F-algebra.
A *co-multiplication* on H is then an algebra homomorphism $\mu\colon H \to H \otimes_F H$ that is *co-associative*, i.e., the diagram

$$\begin{array}{ccc} H & \xrightarrow{\mu} & H \otimes_F H \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle 1\otimes\mu} \\ H \otimes_F H & \xrightarrow[\mu\otimes 1]{} & H \otimes_F H \otimes_F H \end{array}$$

is commutative.

Example: On the polynomial ring $F[x]$, we can define several co-multiplications.

(a) $\mu(x) = x \otimes 1$

(b) $\mu(x) = x \otimes x$

(c) $\mu(x) = x \otimes 1 + 1 \otimes x$

Example: Let G be a group. On the group algebra FG we can define a co-multiplication by

$$\mu(\sigma) = \sigma \otimes \sigma$$

for $\sigma \in G$.

A co-multiplication is *co-commutative* if the diagram

$$\begin{array}{ccc} H & & \\ {\scriptstyle\mu}\big\downarrow & \searrow^{\mu} & \\ H \otimes_F H & \xrightarrow[x\otimes y \mapsto y \otimes x]{} & H \otimes_F H \end{array}$$

is commutative.

A *co-unit* for μ is an algebra homomorphism $\varepsilon\colon H \to F$ such that the diagrams

$$\begin{array}{ccc} H & \xrightarrow{\mu} & H \otimes_F H \\ & {\scriptstyle x\mapsto 1\otimes x}\searrow & \downarrow{\scriptstyle \varepsilon\otimes 1} \\ & & F \otimes_F H \end{array} \quad \text{and} \quad \begin{array}{ccc} H & \xrightarrow{\mu} & H \otimes_F H \\ & {\scriptstyle x\mapsto x\otimes 1}\searrow & \downarrow{\scriptstyle 1\otimes \varepsilon} \\ & & H \otimes_F F \end{array}$$

are commutative.

Thus, if $\mu(x) = \sum_i a_i \otimes b_i$, then

$$x = \sum_i \varepsilon(a_i) b_i = \sum_i a_i \varepsilon(b_i)$$

It follows that μ must be injective.

Example: For our co-multiplications on $F[x]$, we get:

(a) $\mu(x) = x \otimes 1$ has no co-unit.

(b) $\mu(x) = x \otimes x$ has a co-unit given by $\varepsilon(x) = 1$.

(c) $\mu(x) = x \otimes 1 + 1 \otimes x$ has a co-unit given by $\varepsilon(x) = 0$.

Example: The co-multiplication on FG has a co-unit given by

$$\varepsilon(\sigma) = 1$$

for $\sigma \in G$.

The co-multiplication uniquely determines a co-unit:
If ϵ is also a co-unit, then

$$\epsilon(x) = \epsilon\big(\sum_i \varepsilon(a_i)b_i\big) = \sum_i \varepsilon(a_i)\epsilon(b_i) = \varepsilon\big(\sum_i a_i\epsilon(b_i)\big) = \varepsilon(x)$$

Let $\mathfrak{A}$ be an F-algebra.
We then define the *convolution product* on $\mathrm{Hom}_F(H, \mathfrak{A})$ by

$$\varphi * \psi = m \circ (\varphi \otimes \psi) \circ \mu$$

where $m \colon \mathfrak{A} \otimes_F \mathfrak{A} \to \mathfrak{A}$ is the multiplication map.

Thus, if $x \in H$ with $\mu(x) = \sum_i a_i \otimes b_i$, then

$$(\varphi * \psi)(x) = \sum_i \varphi(a_i)\psi(b_i).$$

The convolution product is associative, and $\varepsilon \colon H \to F \to \mathfrak{A}$ is the identity element.
In particular, $\mathrm{Hom}_F(H, \mathfrak{A})$ is an algebra.

An *antipode* on H is an inverse ι of the identity map in $\mathrm{Hom}_F(H, H)$, with respect to the convolution product. I.e., it is a linear transformation $\iota\colon H \to H$ with

$$\varepsilon(x) = \sum_i a_i \iota(b_i) = \sum_i \iota(a_i) b_i$$

when $\mu(x) = \sum_i a_i \otimes b_i$.
An antipode is of course unique.

Example: On $F[x]$, we find that

(b) $\mu(x) = x \otimes x$ has no antipode;

(c) $\mu(x) = x \otimes 1 + 1 \otimes x$ has an antipode given by

$$\iota\big(f(x)\big) = f(-x)$$

Example: On FG, we get an antipode by

$$\iota(\sigma) = \sigma^{-1}$$

for $\sigma \in G$.

Definition: An F-algebra H with a co-multiplication, co-unit and antipode is a *Hopf algebra*.

Example: The polynomial ring $F[x]$ is a Hopf algebra with co-multiplication

$$\mu(x) = x \otimes 1 + 1 \otimes x,$$

co-unit

$$\varepsilon(x) = 0$$

and antipode

$$\iota(x) = -x$$

Example: The group algebra FG is a Hopf algebra with co-multiplication

$$\mu(\sigma) = \sigma \otimes \sigma,$$

co-unit

$$\varepsilon(\sigma) = 1$$

and antipode

$$\iota(\sigma) = \sigma^{-1}$$

Example: Let G be a finite group, and let $F^{(G)}$ denote the ring of functions $G \to F$, with point-wise addition and multiplication. We write the elements as $\sum_{\sigma \in G} a_\sigma e_\sigma$, where the e_σ's are orthogonal idempotents with sum 1. It is then a Hopf algebra with co-multiplication

$$\mu(e_\sigma) = \sum_{\rho\tau=\sigma} e_\rho \otimes e_\tau,$$

co-unit

$$\varepsilon(e_\sigma) = \delta_{\sigma,1}$$

and antipode

$$\iota(e_\sigma) = e_{\sigma^{-1}}$$

Example: Let F be a field of characteristic $\neq 2$, and let $D \in F$ be an element that is not a square. Then we have a Hopf algebra structure on $F \oplus F \oplus F(\sqrt{D}) = Fe_1 \oplus Fe_2 \oplus Fe_3 \oplus Fu$ by

$$\begin{aligned}
\mu(e_1) &= e_1 \otimes e_1 + e_2 \otimes e_2 + \tfrac{1}{2} e_3 \otimes e_3 + \tfrac{1}{2D} u \otimes u, \\
\mu(e_2) &= e_1 \otimes e_2 + e_2 \otimes e_1 + \tfrac{1}{2} e_3 \otimes e_3 - \tfrac{1}{2D} u \otimes u, \\
\mu(e_3) &= (e_1 + e_2) \otimes e_3 + e_3 \otimes (e_1 + e_2), \\
\mu(u) &= u \otimes (e_1 - e_2) + (e_1 - e_2) \otimes u
\end{aligned}$$

The co-unit is given by

$$\varepsilon(e_1) = 1, \quad \varepsilon(e_2) = \varepsilon(e_3) = \varepsilon(u) = 0$$

and the antipode is the identity map.

Theorem: Let H be a Hopf algebra.
Then the antipode ι is an algebra anti-endomorphism.
Moreover, $\varepsilon \circ \iota = \varepsilon$, and

$$\mu(\iota(x)) = \sum_i \iota(b_i) \otimes \iota(a_i)$$

when $\mu(x) = \sum_i a_i \otimes b_i$.

Corollary: If H is commutative or co-commutative, then $\iota^2 = 1$.

Proof: Let $x \in H$ with $\mu(x) = \sum_i a_i \otimes b_i$.
If H is commutative, we have

$$(\iota * \iota^2)(x) = \sum_i \iota(a_i)\iota^2(b_i) = \iota\big(\sum_i a_i \iota(b_i)\big) = \iota(\varepsilon(x)) = \varepsilon(x).$$

If H is co-commutative, we have $\mu(x) = \sum_i b_i \otimes a_i$ as well, and so

$$(\iota * \iota^2)(x) = \sum_i \iota(b_i)\iota^2(a_i) = \iota\big(\sum_i \iota(a_i) b_i\big) = \iota(\varepsilon(x)) = \varepsilon(x).$$

Either way, $\iota * \iota^2 = \varepsilon$, and so $\iota^2 = 1$. ✓

Example: Let $H = F[x, y, y^{-1}]$, where $yx = qxy$ for some $q \in F \setminus \{0, 1\}$.
It is then a Hopf algebra with co-multiplication

$$\mu(x) = x \otimes 1 + y \otimes x, \quad \mu(y) = y \otimes y,$$

co-unit

$$\varepsilon(x) = 0, \quad \varepsilon(y) = 1$$

and antipode

$$\iota(x) = -y^{-1}x, \quad \iota(y) = y^{-1}.$$

H is neither commutative nor co-commutative, and since $\iota^2(x) = q^{-1}x$ we have $\iota^2 \neq 1_H$.