Peaceman Model for Well-Block and Steady-State Einstein Paradigm of Brownian Motion

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Abstract

- Used Einstein Paradigm to create model of steady-state flow to a well in circular reservoir.
- Compared this model to Peaceman's model of flow in a well-block reservoir by using Darcy's Law for rate of production q in our model and equating it to finite discrete approximation of Peaceman's rate of production q.
- Then used Forchheimer equation for non-linear model of flow, found rate of production $\tilde{q}$ and compared it to Peaceman's q through same method.

Einstein Paradigm Model of Flow

Our model of flow, where $u(x, y)$ is "number of particles per unit volume", has this system of equations:

$$\begin{cases} \Delta u = 0 \\ u|_{\partial B(0,R_e)} = P_e \\ u|_{\partial B(0,R_w)} = P_w \end{cases} \qquad \Omega = B(0, R_e)\backslash B(0, R_w) \tag{1}$$

Where Δ is the laplace. The solution $u(r)$ of (1) is:

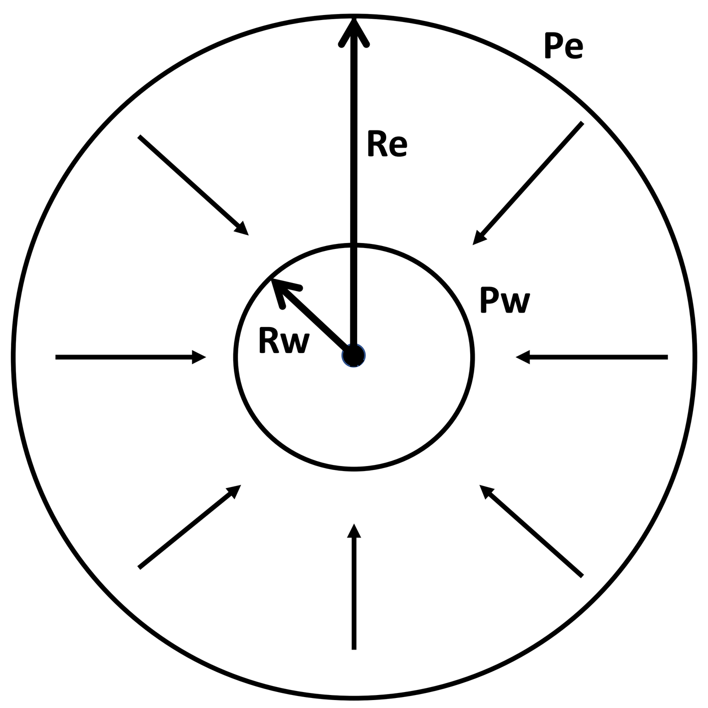
$$u(r) = c_1 \ln{(r)} + c_2 \tag{2}$$

With following definitions:

$$r = \sqrt{x^2 + y^2}\ ,\ c_1 = \frac{P_e - P_w}{\ln{(\frac{R_e}{R_w})}}$$

$$c_2 = \frac{P_e \ln{(\frac{1}{R_w})}}{\ln{(\frac{R_e}{R_w})}} + \frac{P_w \ln{(R_e)}}{\ln{(\frac{R_e}{R_w})}}$$

Circular Reservoir Model



FDA of Peaceman Model

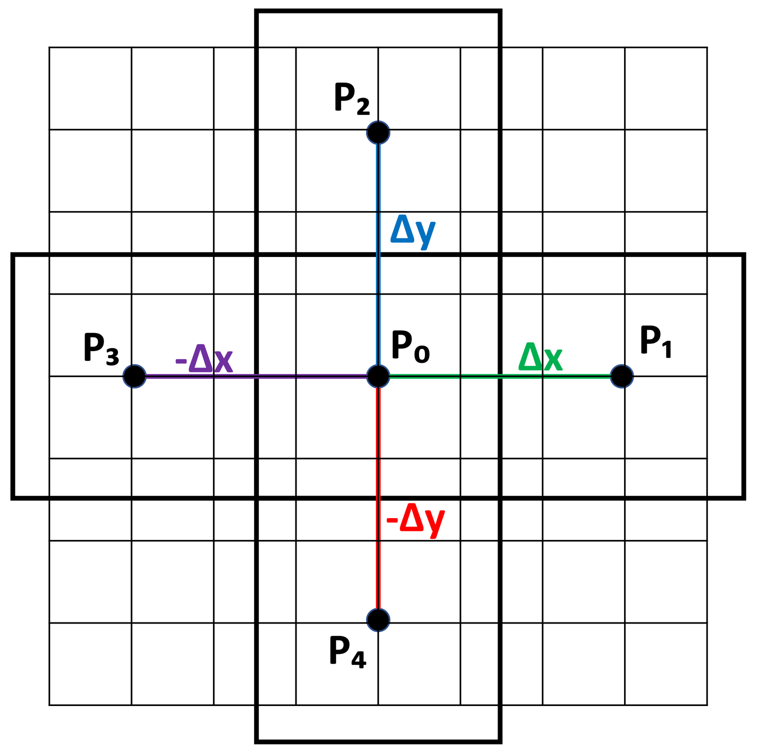
Peaceman's equation for steady-state pressure distribution:

$$q_{i,j} = \frac{kh\Delta y}{\mu \Delta x}(p_{i+1,j} - 2p_{i,j} + p_{i-1,j}) + \\ + \frac{kh\Delta x}{\mu \Delta y}(p_{i,j+1} - 2p_{i,j} + p_{i,j-1}) \tag{3}$$

Where $q_{i,j} = 0$ if $i \neq 0$ or $j \neq 0$. We let $i = 0$, $j = 0$, $h = 1$, and fix $\Delta x = \Delta y$:

$$q = \frac{k}{\mu} \cdot [P_1 + P_2 + P_3 + P_4 - 4P_0] \tag{4}$$

Peaceman Model on Grid



Analytical vs. Discrete Formulae

We have $P_1 = P_2 = P_3 = P_4 = P_e$. We then let $P_w = P_0$, $R_e = \Delta$, and $R_w = R_0$, for which R_0 has an unknown value.
To find the value for R_0, we equate Peaceman's q to our q:

$$\frac{k}{\mu} \cdot [4P_e - 4P_0] = \frac{2\pi k}{\mu} \cdot \frac{P_e - P_0}{\ln{(\frac{\Delta}{R_0})}} \tag{5}$$

Which simplifies to:

$$R_0 = \Delta x \cdot e^{-\pi/2} \tag{6}$$

Exactly what Peaceman derived in his paper, thus our linear steady-state model is justified.

Analytical Solution from Darcy Law

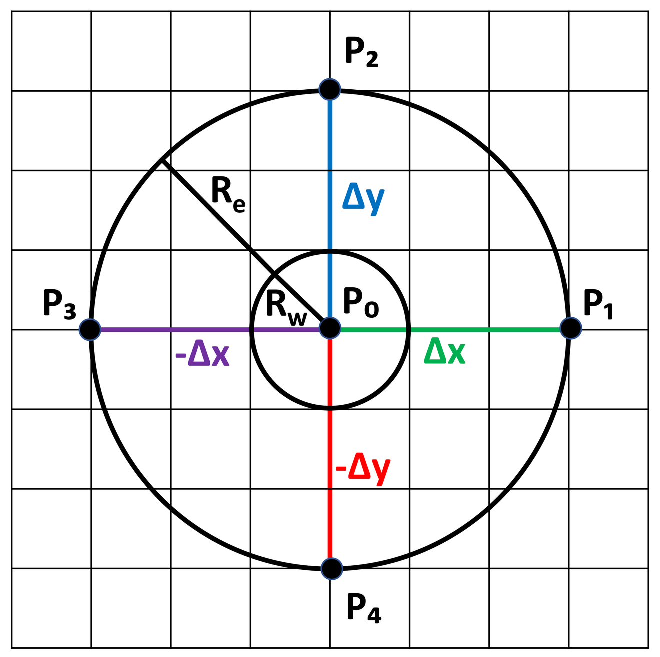
The rate of production q in our model is given:

$$q = \frac{kh}{\mu} \int_{\partial B(0,R_w)} \frac{\partial u}{\partial r} ds \tag{7}$$

Where k is permeability, μ is viscosity, and h is the height of the reservoir. For our calculations, we will assume k, μ to be constants and $h = 1$. We simplify:

$$q = \frac{2\pi k}{\mu} \cdot \frac{P_e - P_w}{\ln{(\frac{R_e}{R_w})}} \tag{8}$$

Circular Model on grid



Forchheimer Relation for Pressure

The generalized Forchheimer equation for rate of production $\tilde{q}$ with equation of continuity $v_r = C\frac{1}{r}$:

$$\tilde{q} = V_r\Big|_{r=R_w} \cdot 2\pi R_w \tag{9}$$

$$\tilde{q} = 2\pi\left(\frac{B - \sqrt{B^2 + 4AD}}{2A}\right) \tag{10}$$

References

[1] A. Einstein. Investigations on the theory of the brownian movement. 1926.

[2] A. et. al. Productivity index for darcy and pre-/post-darcy flow (analytical approach). 2017.

[3] H. Huang and J. Ayoub. Applicability of the forchheimer equation for non-darcy flow in porous media. 2006.

[4] D. Peaceman. Interpretation of well-block pressures in numerical reservoir simulation. 1978.

[5] A. Prada and F. Civan. Modification of darcy's law for the threshold pressure gradient. 1999.

Forchheimer Model of Flow in Media

From Forchheimer's equation in the radial case:

$$-\frac{\partial p}{\partial r} = \alpha v_r(r) + \beta\,(v_r(r))^2 \tag{11}$$

BVP for radial $P(r)$ in the domain with boundary conditions and equation of continuity $v_r = C\frac{1}{r}$:

$$\begin{cases} -\frac{\partial p}{\partial r} = C\frac{\alpha}{r} + C^2\beta\frac{1}{r^2} \\ p|_{r=R_w} = P_w \\ p|_{r=R_e} = P_e \end{cases} ,\quad R_w < r < R_e \tag{12}$$

With solution:

$$P(r) = -C\alpha \ln(\frac{r}{R_e}) + C^2\beta\frac{1}{r} + C_1 \tag{13}$$

With the following definitions and functions:

$$C = \frac{B - \sqrt{B^2 + 4AD}}{2A}$$

$$C_1 = P_e - \left(\frac{B + \sqrt{B^2 + 4AD}}{2A}\right)\beta\frac{1}{R_e}$$

$$A(R_e, R_w) = -\beta\left(\frac{R_e - R_w}{R_e \cdot R_w}\right)$$

$$B(R_e, R_w) = \alpha \ln\left(\frac{R_w}{R_e}\right)$$

$$D(P_e, P_w) = P_e - P_w$$

Results: Forchheimer vs. Peaceman

By equating our $\tilde{q}$ and Peaceman's q, we can get corresponding value of R_w with fixed β:

$$2\pi\left(\frac{B - \sqrt{B^2 + 4AD}}{2A}\right) = \frac{k}{\mu}[4P_e - 4P_0] \tag{14}$$

- Let $R_e = \Delta$ in functions $A(R_e, R_w)$ and $B(R_e, R_w)$. Let $P_w = P_0$ in function $D(P_e, P_w)$
- α and β are given from the experiment.

End goal is to find $R_w = R_0$ which will solve transcendent equation (14) with respect to known values of $\alpha,\ \beta,\ P_0,\ P_e, \Delta$.