Finite Speed of Propagation in One Dimensional Degenerate Einstein Brownian Motion Model

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Abstract

We use the generalization of Einstein's paradigm of Brownian motion for diffusion when the parameter of the time interval of *free jump* degenerates to derive a system of one dimensional degenerate nonlinear partial differential equations. The solution of the system represents the number of particles per unit volume during the diffusion process as the time interval of free jumps degenerates. Specifically, in these equations, the time interval depends on the solution of the equations. We will demonstrate the finite speed of propagation of the system by using the construction of Christov-Hevage-Ibraguimov-Islam and the subsequent methods of Kompaneets–Zel'dovich–Barenblatt. With the finite speed of propagation(the localization property) being defined as: If $u(x_0,0)>0$ on $|x+x_0|\le\delta=$ constant and $u(x_0,0)\equiv 0$ for $|x+x_0|>\delta$. Then, u(s,t)=0 for $|s-x_0|\gg\delta$ and $0\le t\le T$.

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Einstein's paradigm in one dimension

Let $u(x,t)$ be the observed density function at $x\in\mathbb{R}$ and time t .Let τ be the time interval of particles' "free jumps"(jumps of particles without "collision") with length $\Delta\in\mathbb{R}$. Let us define $\varphi(\Delta)$ as the frequency of "free jumps" with following assumptions.

- Symmetric constraint :

$$\varphi(\Delta)=\varphi(-\Delta) \tag{1}$$

- " Completeness " of the universe of all possible free jumps:

$$\int_{\mathbb{R}}\varphi(\Delta)d\Delta=1 \tag{2}$$

- Expectation of free jumps:

$$\int_{-\infty}^{\infty}\Delta\varphi(\Delta)d\Delta=0 \tag{3}$$

- Einstein Conservation Law

$$u(x,t+\tau)=\int_{-\infty}^{\infty}u(x+\Delta,t)\varphi(\Delta)d\Delta \tag{4}$$

All Resulting in the derivation of the classical heat equation:

$$L_0u=\tau\frac{\partial u}{\partial t}-D_0\frac{\partial^2 u}{\partial x^2}=0;\quad D_0=\frac{\int_{-\infty}^{\infty}\Delta^2\varphi(\Delta)d\Delta}{2} \tag{5}$$

One Dimensional Model

For our project, the situation where τ is non-constant is of interest. A non-constant τ does not necessarily guarantee an infinite speed of propagation.

We will use,

$$\tau=\frac{1-\alpha}{u^{\alpha}} \tag{6}$$

Here $0<\alpha<1$.

Let $u>0$ be a classical solution of the following system.

$$L_0u=\frac{\partial u^{1-\alpha}}{\partial t}-\frac{\partial^2 u}{\partial x^2}=0 \text{ in the domain } 0<x<2L,\ 0<t<\infty, \tag{7}$$

$$u(x,0)=u_0(x)>0 \text{ for } 0<x<\frac{L}{2}, \tag{8}$$

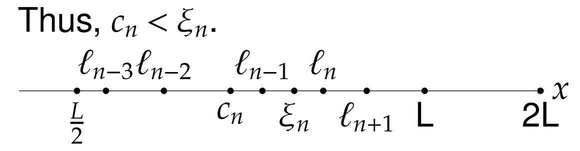
$$u(x,0)=0 \text{ for } \frac{L}{2}\le x<2L, \tag{9}$$

$$\frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(2L,t)=0 \text{ for } 0<t<\infty. \tag{10}$$

Let $\ell_n=L-2^{-n}$ and $I_n=\{x:2L>x>\ell_n\}$ and $J_n=I_n\setminus I_{n+1}$.

In the layers J_{n-1} and J_{n-2} introduce auxiliary points $\xi_n\in J_{n-1}$ and $c_n\in J_{n-2}$.

Thus, $c_n<\xi_n$.



Goal: Prove that u preserves the Strong Maximum Principle.

Generalization of Weak Maximum Principle

Lemma 1

Assume for some generic w(x,t) on $\Omega_T=\Omega\times[0,T]$

$$a(x,t)\frac{\partial w}{\partial t}-\frac{\partial^2 w}{\partial x^2}<0. \tag{11}$$

If $a(x,t)\ge 0$ then $w(x,t)\le\max_{\Gamma(\Omega_T)}w(x,t)$

Lemma 2

Let $u(x,t)$ be a solution of the inequality

$$u^{-\alpha}\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}\le 0 \tag{12}$$

with $0<\alpha<1$.

If $u(x,t)\ge 0$ then $u(x,t)\le\max_{\Gamma(\Omega_T)}u(x,t)$

Iterative Procedure

The following lemma can be obtained by integrating (7) with respect to τ.

Lemma 3

The function

$$v(x)=\int_0^T u(x,\tau)d\tau \tag{13}$$

satisfies the maximum principle such that

$$v(c_n)\ge v(\xi_n),\text{for any } c_n\le\xi_n. \tag{14}$$

The next lemma can be obtained through integrating (7) with respect to τ again, integrating with respect to x over I_{n-1}, and then applying the MVT.

Lemma 4

If $u(x,t)$ is the solution to the IBVP, (7) - (10),

then

$$\int_{I_n}u^{1-\alpha}(x,T)dx\le 2^{n+1}\int_0^T u(\ell_{n-1},\tau)d\tau \tag{15}$$

Subsequently through the integral MVT,

Lemma 5

$$\int_{I_n}u^{1-\alpha}(x,T)dx\le 2^{2n}\int_0^T\int_{I_{n-2}}u(x,\tau)dxd\tau \tag{16}$$

The following theorem can be achieved through the use of the MVT, the maximum principle of $v(x)$, and Hölder's Inequality.

Theorem 6

Assume that $u_0(x)$ in (8) is such that

$$u(x,t)\le 1. \tag{17}$$

Let

$$\tilde{I}_n\triangleq\max_{0<\tau<T}\int_{I_n}u^{1-\alpha}(x,\tau)dx. \tag{18}$$

Then

$$\tilde{I}_n\le 2^{3n-2}C(\alpha)T\,\tilde{I}_{n-3}^{1+\epsilon}. \tag{19}$$

Here

$$\epsilon=\frac{\alpha}{1-\alpha} \text{ and } C(\alpha)=(2L)^{\frac{1-2\alpha}{1-\alpha}} \text{ in case when } \alpha<\frac{1}{2}, \tag{20}$$

and

$$\epsilon=1 \text{ and } C(\alpha)=1 \text{ in case when } \alpha\ge\frac{1}{2}. \tag{21}$$

Localization Property

Finally, via the Ladyženskaja iterative Lemma (see Ladyženskaja, Solonnikov, and Ural'ceva, 1968, Chap. II §5), the following theorem is established.

Theorem 7

Assume the solution u(x,t) is s.t.

$$\tilde{I}_0\le(C(\alpha))^{-\frac{1}{\epsilon}}8^{-\frac{1}{\epsilon^2}}T^{-1} \tag{22}$$

Then,

$$\tilde{I}_n(T)\to 0\ as\ n\to\infty \tag{23}$$

So,

$$\tilde{I}_n=\max_{0<\tau<T}\int_{I_n}u^{1-\alpha}(x,\tau)dx\to 0 \tag{24}$$

Thus,

$$\tilde{I}_n\to 0\Rightarrow u^{1-\alpha}(x,t)\equiv 0;t\in[0,T],x\in[L,2L] \tag{25}$$

Constants above,$\epsilon>0$ and $C(\alpha)$, are the same as in Theorem 7.

References

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