Traveling Band Model for E. coli Transport in the Presence of Limited Immovable Food using the Einstein Paradigm of Brownian Motion

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Introduction

Keller and Segel's work about the chemotaxis of E. coli bacteria as they consume sugar is an example of a predator-prey model where the bacteria are the mobile predators moving in traveling bands towards their food and the sugar is their immovable prey. In their scenario shown in the figure below, E. coli bacteria are located in a petri dish to the left of a tube filled with an unlimited amount of sugar acting as food for the bacteria.

We will model a similar scenario as described in Keller-Segel, but we will obtain our PDE model using the Einstein paradigm of Brownian motion. Additionally, we will assume that there is a limited amount of food available in the tube.

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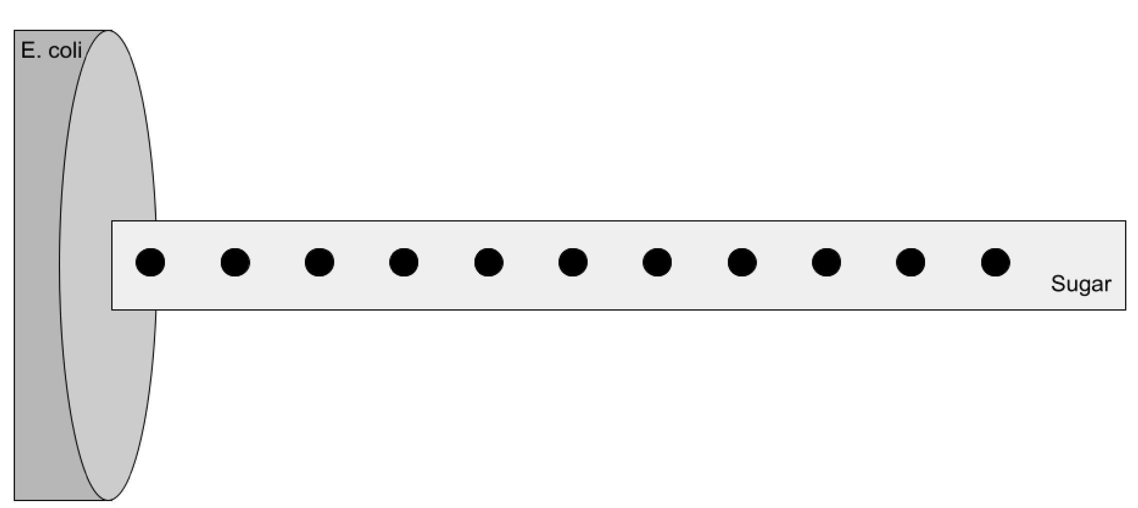


Variables and Einstein Paradigm

t: time
x: position
$w(x,t)$: concentration of bacteria per unit volume
$v(x,t)$: concentration of food per unit volume
Δ: length of the free jump of the bacteria before collision
τ: time interval of the free jump before collision
$\varphi(\Delta)$: frequency of the free jumps of the bacteria of size Δ

$$w(x,t+\tau)=\int_{-\infty}^{\infty}w(x+\Delta,t)\cdot\varphi(\Delta)\,d\Delta+\int_t^{t+\tau}f(x,s)\,ds \qquad (1)$$

where $w(x+\Delta,t)\cdot\varphi(\Delta)$ represents the weighted free jump of the bacteria and $f(x,s)$ represents the crowd transport of the bacteria.

Hypotheses of Bacteria Behavior

$$\tau\frac{\partial w}{\partial t}=\Delta_e^w\frac{\partial w}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 w}{\partial x^2}+\int_t^{t+\tau}f(x,s)\,ds \qquad (2)$$

$\tau=1$ and $\sigma^2=\mu$, where μ is the motility coefficient (3)

Expected Value of Free Jump (Δ_e^w)

$\Delta_e^w=-\beta\dfrac{\partial\ln(v)}{\partial x}$, where β is a constant (4)

Crowd Transport

$\displaystyle\int_t^{t+\tau}f(x,s)\,ds=w\frac{\partial\Delta_e^w}{\partial x}=-\gamma w\frac{\partial^2\ln(v)}{\partial x^2}$, with $\gamma=\beta$ (5)

Hypotheses of Food Consumption

$$\tau\frac{\partial v}{\partial t}=\Delta_e^v\frac{\partial v}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial x^2}+\int_t^{t+\tau}g(x,s)\,ds \qquad (6)$$

$\Delta_e^v=0$ and $\sigma^2=D$, where D is the diffusion rate (7)

Consumption of Limited Food by the Bacteria

$$\int_t^{t+\tau}g(x,s)\,ds=-k\tau wv, \qquad (8)$$

where k is the consumption rate of the food by the bacteria

Diffusion

Keller and Segel assume that the effect of the diffusion rate of the food on its concentration is negligible when compared to the effect of its consumption by the bacteria, letting $D=0$.

PDE Model

$$\frac{\partial w}{\partial t}=-\gamma\frac{\partial}{\partial x}\left(w\frac{\partial\ln(v)}{\partial x}\right)+\frac{1}{2}\mu\frac{\partial^2 w}{\partial x^2} \qquad (9)$$

$$\frac{\partial v}{\partial t}=-kwv \qquad (10)$$

Traveling Band Structure

$\zeta=x-ct$, where c is the constant speed of the band (11)

$$\frac{\partial w}{\partial t}=-cw' \qquad \frac{\partial w}{\partial x}=w' \qquad \frac{\partial^2 w}{\partial x^2}=w'' \qquad (12)$$

$$\frac{\partial v}{\partial t}=-cv' \qquad \frac{\partial v}{\partial x}=v' \qquad \frac{\partial^2 v}{\partial x^2}=v'' \qquad (13)$$

ODE Model

$$cw'=\gamma\left(w(\ln v)'\right)'-\frac{1}{2}\mu w'' \qquad (14)$$

$$cv'=kwv \qquad (15)$$

Closed Form Solutions

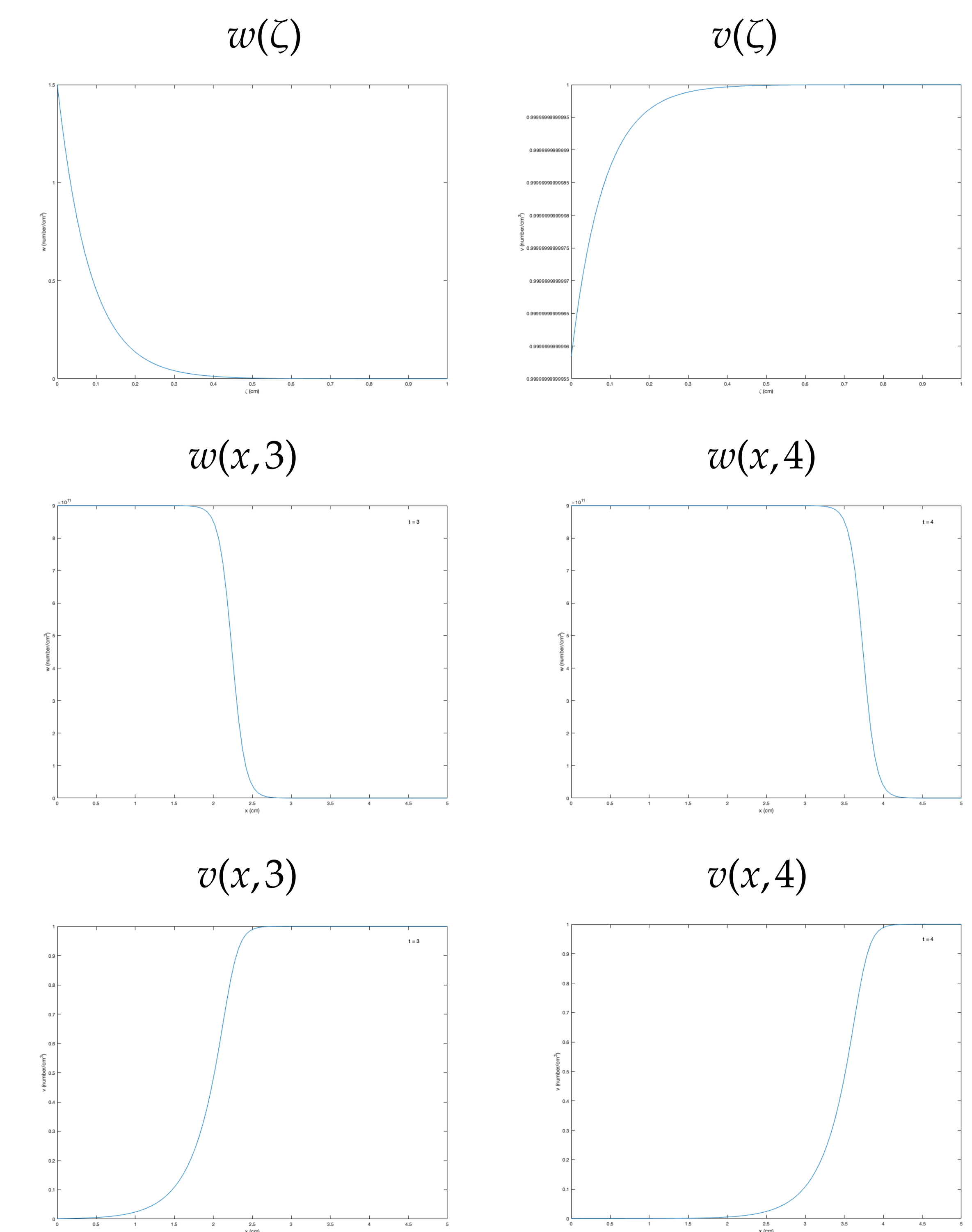
$$w=\frac{c}{Qe^{\frac{2c}{\mu}\zeta}+\frac{\gamma k}{c}} \qquad (16)$$

$$v=\widetilde{Q}\left(Q+\frac{\gamma k}{c}e^{\frac{-2c}{\mu}\zeta}\right)^{-\frac{\mu}{2\gamma}} \qquad (17)$$

Parameters

Estimates based on Keller-Segel		
Symbol	Value	Definition
c	1.5	constant band speed
Q	1	constant of integration
$\widetilde{Q}$	1	constant of integration
μ	0.25	motility coefficient
γ	0.5	proportionality coefficient of Δ_e
k	5×10^{-12}	food consumption rate

Graphs



Conclusion

A predator-prey relationship with limited food results in the depletion of the food and in a saturation of the predator up to a certain carrying capacity. Our model can be applied to a variety of movable or immovable food scenarios.

References

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