Math 2450: Unit Tangent, Principal Unit Normal Vectors, and Curvature Reference Sheet

Unit Tangent Vector and Principal Unit Normal Vector

If $\mathbf{R}(t)$ is a vector function that defines a smooth graph, then at each point a **unit tangent** is

$$\mathbf{T}(t) = \frac{\mathbf{R}'(t)}{\|\mathbf{R}'(t)\|}$$

and the **principal unit normal** vector is

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

Note: These are unit vectors that change direction depending on where you are on the curve.

Arc Length Function

Let C be a piecewise-smooth curve that is the graph of the vector described parametrically by $\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, and let $P_0 = P(t_0)$ be a particular point on C (called the **base point**). Then the length of C from the base point P_0 to the variable point $P(t)$ is given by the **arc length function** $s(t)$ defined by

$$s(t) = \int_{t_0}^{t} \|\mathbf{R}'(u)\|\, du = \int_{t_0}^{t} \sqrt{(\tfrac{dx}{du})^2 + (\tfrac{dy}{du})^2 + (\tfrac{dz}{du})^2}\ \ du$$

Note: The function $s(t)$ measures the distance along the curve C from $P(t_0)$ to $P(t)$ when $t > t_0$ OR from $P(t)$ to $P(t_0)$ when $t < t_0$. This interpretation of arc length is best for describing a curve's geometric features.

Speed (As the Derivative of Arc Length)

Suppose an object moves along a smooth curve C that is the graph of the position function $\mathbf{R}(t) = \langle x(t), y(t), z(t)\rangle$, where $\mathbf{R}'(t)$ is continuous on the interval $[t_1, t_2]$. Then the object has **speed**

$$\frac{ds}{dt} = \|\mathbf{V}(t)\| = \|\mathbf{R}'(t)\| \qquad \text{for} \quad t_1 \leq t \leq t_2$$

T and N Expressed in Terms of Arc Length

If $\mathbf{R}(t)$ has a piecewise-smooth graph and is represented as $\mathbf{R}(s)$ in terms of the arc length parameters, then the unit tangent vector $\mathbf{T}$ and the principal unit normal vector $\mathbf{N}$ satisfy

$$\mathbf{T} = \frac{d\mathbf{R}}{ds} \qquad \text{and} \qquad \mathbf{N} = \frac{1}{k}\frac{d\mathbf{T}}{ds}$$

where $k = \left\|\frac{d\mathbf{T}}{ds}\right\|$ is a scalar function of s.

Curvature Formulas

Type	Given Information	Formula
Arc Length Parameter	$\mathbf{R}(s)$	$\left\lVert \frac{d\mathbf{T}}{ds} \right\rVert$
Two-Derivatives Form	$\mathbf{R}(t)$	$\frac{\lVert \mathbf{T}'(t) \rVert}{\lVert \mathbf{R}'(t) \rVert}$
Cross-Derivative Form	$\mathbf{R}(t)$	$\frac{\lVert \mathbf{R}' \times \mathbf{R}'' \rVert}{\lVert \mathbf{R}' \rVert^3}$
Functional Form	$y = f(x)$	$\frac{\lvert f''(x) \rvert}{(1+[f'(x)]^2)^{\frac{3}{2}}}$
Parametric Form	$x = x(t), \quad y = y(t)$	$\frac{\lvert x'y'' - y'x'' \rvert}{[(x')^2+(y')^2]^{\frac{3}{2}}}$
Polar Form	$r = f(\theta)$	$\frac{\left\lvert r^2+2(r')^2-rr'' \right\rvert}{(r^2+(r')^2)^{\frac{3}{2}}}$