Math 2450: Vectors Formula Reference Sheet

What is a vector? A vector is a quantity that has both magnitude and direction. Vectors are represented by a directed line segment (or arrow) with an **initial point** P and **terminal point** Q, which is written as **PQ** or $\overrightarrow{PQ}$. Be careful, order does matter in the expression of a vector, as **QP** is a vector with initial point Q and terminal point P.

Why do we use vectors? Vectors are used to represent force, velocity, acceleration, and momentum. Vectors in $\mathbb{R}^3$ are used to describe motion in space more efficiently.

Standard Components of a Vector

In $\mathbb{R}^3$: If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are points in a coordinate plane, then

$P_1P_2 = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ and $P_2P_1 = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$

Note: Order **does** matter: It is the terminal point coordinate minus the corresponding initial point coordinate.

Standard Representation of a Vector

Vectors may be expressed as a linear combination of standard basis vectors:

In $\mathbb{R}^3$: Standard basis vectors: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1 \langle 1, 0, 0 \rangle + v_2 \langle 0, 1, 0 \rangle + v_3 \langle 0, 0, 1 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

Note: The v_1 must always be attached with the **i**, v_2 with the **j**, and v_3 with the **k**.

Vector Operations

Addition: $\langle a, b \rangle + \langle c, d \rangle = \langle a + c, b + d \rangle$

Subtraction: $\langle a, b \rangle - \langle c, d \rangle = \langle a - c, b - d \rangle$

Scalar Multiple: $k \cdot \langle a, b \rangle = \langle k \cdot a, k \cdot b \rangle$

Properties

$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ $\quad$ $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

$\mathbf{u} + \mathbf{0} = \mathbf{u}$ $\quad$ $\mathbf{u} + (\mathbf{-u}) = \mathbf{0}$

$s(\mathbf{u} + \mathbf{v}) = s\mathbf{u} + s\mathbf{v}$ $\quad$ $(s + t)\mathbf{u} = s\mathbf{u} + t\mathbf{u}$

$(st)\mathbf{u} = s(t\mathbf{u})$

Note: Vector operations are similar in $\mathbb{R}^3$. s, t are scalars and **u, v, w** are vectors in the plane. Two vectors are **parallel** if they are scalar multiples of one another.

Magnitude of a Vector

The magnitude of a vector is its length and is denoted by $\|\mathbf{PQ}\|$.

In $\mathbb{R}^3$: $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, $\|\mathbf{v}\| = \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2}$

Note: The magnitude is a value (scalar) and is **not** an absolute value! For example:
Given $\mathbf{w} = \langle -1, 2, -3 \rangle$: the magnitude of $\mathbf{w}$, $\|\mathbf{w}\| = \sqrt{(-1)^2 + (2)^2 + (-3)^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$
whereas the absolute value of $\mathbf{w}$, $|\mathbf{w}| = |\langle -1, 2, -3 \rangle| = \langle |-1|, |2|, |-3| \rangle = \langle 1, 2, 3 \rangle$.

A **unit vector u** has a magnitude of 1 and is in the direction of a given vector $\mathbf{v}$: $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$

Equation of a Sphere

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^3$$

is a sphere with center (a, b, c) radius r.

Equation of a Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z^2$$

Equation of a Paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$$

Distances Between Points in $\mathbb{R}^3$

The distance $|P_1P_2|$ between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Given vectors $\mathbf{v}= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{w}= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

Dot Product

The dot product $(\mathbf{v} \cdot \mathbf{w})$ is given by

$$\mathbf{v} \cdot \mathbf{w} = a_1b_1 + a_2b_2 + a_3b_3$$

Properties

$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 \qquad c(\mathbf{v} \cdot \mathbf{w}) = (c\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (c\mathbf{w})$$

$$\mathbf{0} \cdot \mathbf{v} = 0 \qquad \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$$

Note: The dot product is also known as the **inner product**. The result of a dot product of vectors is a **scalar**, **not a vector.** The dot product is used to calculate the angle between 3-dimensional vectors. An important application of the dot product and projections is in the calculation of Work: $\mathbf{W} = \mathbf{F} \cdot \mathbf{PQ}$.

Angle Between Two Vectors

If θ is the angle between the nonzero vectors $\mathbf{v}$ and $\mathbf{w}$, then $cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \cdot \|\mathbf{w}\|}$.

We can now define a **Geometrical Formula for the Dot Product**:

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \, \|\mathbf{w}\| \, cos\theta$$

where $\theta \in [0, \pi]$ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

Note: Two vectors $\mathbf{v}$ and $\mathbf{w}$ are **orthogonal** (or perpendicular) if $\mathbf{v} \cdot \mathbf{w} = 0$.

Projections

If $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors, then the **vector projection** of $\mathbf{v}$ in the direction of $\mathbf{w}$ (a vector) is

$$proj_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\mathbf{w}$$

scalar projection of $\mathbf{v}$ onto $\mathbf{w}$ (a number) is

$$comp_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|}$$

Cross Product

If $\mathbf{v}= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{w}= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, the **cross product**, written $\mathbf{v}\times\mathbf{w}$, is the vector

$$\mathbf{v} \times \mathbf{w} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

The definition is found by using the determinant $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Note: The result of taking the cross product (or **outer product**) of two vectors is a vector. An important application of the cross product is the calculation of torque: $\mathbf{T}= \mathbf{PQ} \times \mathbf{F}$

Properties of Cross Product

If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are vectors in $\mathbb{R}^3$ and s, t are scalars, then the following properties can be derived:

$$(s\mathbf{v}) \times (t\mathbf{w}) = st(\mathbf{v} \times \mathbf{w}) \qquad \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$$

$$\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v}) \qquad (\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$$

$$\mathbf{v} \times \mathbf{v} = 0 \qquad \mathbf{v} \times \mathbf{0} = \mathbf{0} \times \mathbf{v} = \mathbf{0}$$

$$\|\mathbf{v} \times \mathbf{w}\|^2 = \|\mathbf{v}\|^2 \, \|\mathbf{w}\|^2 - (\mathbf{v} \cdot \mathbf{w})^2 \qquad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$$

Geometric Interpretation of Cross Product

◦ If **v** and **w** are nonzero vectors in $\mathbb{R}^3$ that are not multiples of one another, then $\mathbf{v} \times \mathbf{w}$ is **orthogonal** to both **v** and **w**.

◦ If **v** and **w** are nonzero vectors in $\mathbb{R}^3$ with θ the angle between **v** and **w** ($0 \le \theta \le \pi$), then $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \, \|\mathbf{w}\| \, sin\theta$

◦ Let **u, v, w** be nonzero vectors that do not all lie in the same plane. Then,

- Area of a parallelogram: $A = \|\mathbf{u} \times \mathbf{v}\|$
- Area of a triangle: $A = \frac{1}{2} \|\mathbf{u} \times \mathbf{v}\|$
- Volume of a parallelepiped: $V = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$

Limit of a Vector Function

Suppose the components f_1, f_2, f_3 of the vector function

$$\mathbf{F}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$$

all have finite limits as $t \to t_0$, where t_0 is any number or ∞ or $-\infty$. Then the **limit** of $\mathbf{F}(t)$ as $t \to t_0$ is the vector

$$\lim_{t \to t_0} \mathbf{F}(t) = \left[\lim_{t \to t_0} f_1(t)\right] \mathbf{i} + \left[\lim_{t \to t_0} f_2(t)\right] \mathbf{j} + \left[\lim_{t \to t_0} f_3(t)\right] \mathbf{k}$$

Note: The limit definition of a vector function satisfies all limit rules [limit of a sum, difference scalar multiple, and product (cross product and/or dot product)].

Derivative of a Vector Function

The vector function **F**(t)= $f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ is **differentiable** whenever to component functions f_1, f_2, f_3 are differentiable, and in this case

$$\mathbf{F}'(t) = f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} + f_3'(t)\mathbf{k}$$

Vector Motion

An object that moves in such a way that its position at time t is given by the vector function **R**(t) is said to have

Position vector, R(t) and **velocity, V**= $\frac{d\mathbf{R}}{dt}$

At any time t,

◦ the **speed** is $||\mathbf{V}||$, the magnitude of velocity,

◦ the **direction of motion** is the unit vector $\frac{\mathbf{V}}{||\mathbf{V}||}$, and

◦ the **acceleration vector** is the derivative of the velocity:

$$\mathbf{A} = \frac{d\mathbf{V}}{dt} = \frac{d^2\mathbf{R}}{dt^2}$$

Vector Integrals

Let **F**(t)= $f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$, where f_1, f_2, f_3 are continuous on the closed interval $a \le t \le b$. Then the **indefinite integral** of **F**(t) is the vector function

$$\int \mathbf{F}(t)dt = [\int f_1(t)dt]\mathbf{i} + [\int f_2(t)dt]\mathbf{j} + [\int f_3(t)dt]\mathbf{k} + \mathbf{C}$$

where $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$ is an arbitrary constant vector. The **definite integral** of **F**(t) on $a \le t \le b$ is the vector

$$\int_a^b \mathbf{F}(t)dt = [\int_a^b f_1(t)dt]\mathbf{i} + [\int_a^b f_2(t)dt]\mathbf{j} + [\int_a^b f_3(t)dt]\mathbf{k} + \mathbf{C}$$