Math 2450: Tangent Planes, Linear Approximations, and Gradients

What is a tangent plane? A tangent plane is a plane that contains the tangent to every smooth curve on a surface S [$z = f(x,y)$] that passes through a given point $P_0(x_0, y_0, z_0)$. Recall that partial derivatives calculate the slope of the tangent line at a particular point. Therefore, the **equation for the tangent plane** requires us to take partial derivatives with respect to x and y:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \qquad [1]$$

Why are we interested in tangent planes? Tangent planes give us a nice way to approximate a surface near a given point. This is often referred to as **incremental (or linear) approximations of a function**. Let $\Delta x = x - x_0$ and $\Delta y = y - y_0$. Since $z_0 = f(x_0, y_0)$, let's rewrite [1] as:

$$z - f(x_0, y_0) = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$
$$\Rightarrow z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + f(x_0, y_0)$$

Therefore, the formula for incremental (linear) approximations is

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + f(x_0, y_0)$$

as long as Δx and Δy are small. Using incremental approximations we can estimate the change of a function and calculate maximum percentage errors.

And note that the formula for linear approximations may be expressed using differentials

$$f(x_0 + dx, y_0 + dy) \approx z = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy + f(x_0, y_0)$$

Tangents planes also have a special relationship with **gradients**. What is a gradient? If we think about partial derivatives in terms of directions, we may see that $f_x(x_0, y_0)$ is the direction in terms of the unit vector $\mathbf{i}$ (x-direction) and $f_y(x_0, y_0)$ is the direction in terms of the unit vector $\mathbf{j}$ (y-direction). The **gradient** (∇f) of a differentiable function f is a vector given by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

and the gradient evaluated at a specific point is

$$\nabla f(x_0, y_0) = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}$$

Let S be a level surface defined by the function in three variables $f(x, y, z) = k$ (k a constant). Then if $P_0(x_0, y_0, z_0)$ is a point on S, the gradient of f at P_0 [$\nabla f(x_0, y_0, z_0)$] is **perpendicular (or normal) to the tangent plane** surface at P_0. In other words, the gradient allows us to find a vector (and even a line) that is perpendicular to a level surface!

Tangent Plane Formula	$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ Or $Ax + By + Cz + D = 0$
Incremental (Linear) Approximation Formula	$f(x_0 + \Delta x, y_0 + \Delta y) \approx z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + f(x_0, y_0)$ Or $f(x_0 + dx, y_0 + dy) \approx z = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy + f(x_0, y_0)$
Gradient Formula	$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$