Math 2450: Potential Functions

We saw (in a previous resource) that if we know the vector field $\mathbf{F}$ is conservative, then $\int_C \mathbf{F} \cdot d\mathbf{R}$ is independent of path. This fact allows us to more easily calculate the line integral as long as we may find a potential function for $\mathbf{F}$. Therefore, we have two tasks to tackle:

1. Determining whether or not a vector field is conservative
2. Given that $\mathbf{F}$ is a conservative vector field, determining a potential function for the vector field

Determining that a Vector Field is Conservative

Given $\mathbf{F}(x,y,z) = f(x,y,z)\mathbf{i} + g(x,y,z)\mathbf{j} + h(x,y,z)\mathbf{k}$:

1. Identify $f(x,y,z)$, $g(x,y,z)$, and $h(x,y,z)$ in $\mathbf{F}$
2. Calculate *curl* $\mathbf{F}$
   (a) *curl* $\mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle f, g, h \rangle$
3. If *curl* $\mathbf{F} = 0$, then the vector field $\mathbf{F}$ is conservative

Determining a Potential Function for a Conservative Vector Field

We must find a potential function f such that $\bigtriangledown f = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} = \mathbf{F}$.

For $\mathbf{F}(x,y,z) = u(x,y,z)\mathbf{i} + v(x,y,z)\mathbf{j} + w(x,y,z)\mathbf{k}$:

Steps	**Example**
Identify the equations: $\frac{\partial f}{\partial x} = u(x,y,z)$ $\quad \frac{\partial f}{\partial y} = v(x,y,z)$ $\quad \frac{\partial f}{\partial z} = w(x,y,z)$	Find a scalar potential function for $\mathbf{F} = (20x^3z + 2y^2)\mathbf{i} + (4xy)\mathbf{j} + (5x^4 + 3z^2)\mathbf{k}$ $\frac{\partial f}{\partial x} = 20x^3z + 2y^2$ $\quad \frac{\partial f}{\partial y} = 4xy$ $\quad \frac{\partial f}{\partial z} = 5x^4 + 3z^2$
Integrate $\frac{\partial f}{\partial x} = u$ with respect to x to get a function of the form: $f(x,y,z) = \int u\ dx + g(y,z)$	$f(x,y,z) =$ $\int 20x^3z + 2y^2 dx = 5x^4z + 2xy^2 + g(y,z)$
Take the partial derivative of $f(x,y,z) = \int u\ dx + g(y,z)$ with respect to y a.) Set $\frac{\partial}{\partial y}[\ \int u\ dx + g(y,z)] = v$ b.) Solve for $\frac{\partial g}{\partial y}$	$\frac{\partial}{\partial y}[5x^4z + 2xy^2 + g(y,z)] = 4xy + \frac{\partial g}{\partial y}$ $4xy + \frac{\partial g}{\partial y} = 4xy$ Therefore, we see $\frac{\partial g}{\partial y} = 0$ and $g(y,z) = h(z)$ So now we have: $f = 5x^4z + 2xy^2 + h(z)$
Take the partial derivative of $f = 5x^4z + 2xy^2 + h(z)$ with respect to z a) Set $\frac{\partial}{\partial z}[5x^4z + 2xy^2 + h(z)] = w$ b) Solve for $h'(z)$	$\frac{\partial}{\partial z}[5x^4z + 2xy^2 + h(z)] = 5x^4 + h'(z)$ $5x^4 + h'(z) = 5x^4 + 3z^2$ Therefore, we see $h'(z) = 3z^2 \implies h(z) = z^3 + C$ **Scalar potential function for F:** $F(x,y,z) = 5x^4z + 2xy^2 + z^3 + C$