Math 2450: Polar, Cylindrical, and Spherical Coordinates

What are polar coordinates? We were first introduced to polar coordinates in Calc II. Recall that polar coordinates replace the typical (x, y) points with (r, θ) points. θ is the angle rotated on the unit circle and r is the number of units we move in the direction of that angle.

Why are polar coordinates important for double integration? We use polar coordinates in double integrals when the integrand or the region of integration have polar forms that are easier to work with than the rectangular forms. For example, the integrals below are equivalent, but the integral on the right is much nicer to integrate.

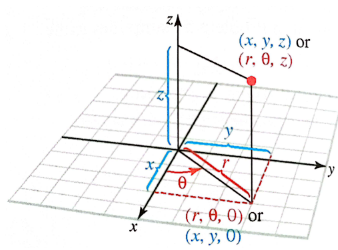
$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2+y^2+1)\ dydx \qquad\qquad \int_0^{2\pi}\int_0^2 (r^2+1)\ rdr\ d\theta$$

Changing Cartesian Integrals to Polar Integrals

1. Substitute x, y, and $dx\ dy$ using the conversion formulas in the given integral $\iint_R f(x, y)\ dA$
2. Convert the region of integration R to polar form D. Thus,

$$\iint_R f(x,y)\ dA = \iint_D f(rcos\ \theta, rsin\ \theta)\ r\ dr\ d\theta$$

What are cylindrical coordinates? Cylindrical coordinates extend polar coordinates to three dimensions ($\mathbb{R}^3$). In cylindrical coordinates, we measure the point in the xy-plane in polar coordinates, with the same z-coordinate as in the Cartesian coordinate system.



Cylindrical coordinates are ideal for representing cylindrical surfaces and surfaces of revolution about the z-axis:

	Cylinder	**Cone**	**Paraboloid**	**Hyperboloid**
Rectangular Equation:	$x^2+y^2=a^2$	$x^2+y^2=z^2$	$x^2+y^2=az$	$x^2+y^2-z^2=1$
Cylindrical Equation:	$r=a$	$r=z$	$r^2=az$	$r^2=z^2+1$

Why are cylindrical coordinates important for triple integration? We use cylindrical coordinates in triple integrals when a two-dimensional region of integration can be described more naturally in terms of polar coordinates than rectangular coordinates.

Triple Integrals in Cylindrical Coordinates Let D be a solid with upper surface $z = v(r, \theta)$ and lower surface $z = u(r, \theta)$, and let A be the projection of the solid onto the xy-plane expressed in polar coordinates. Then, if $f(x, y, z)$ is continuous on D, the triple integral of f over D is

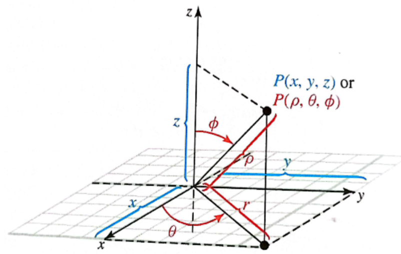
$$\iiint_D f(x,y,z)\ dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{u(r,\theta)}^{v(r,\theta)} f(r\ cos\ \theta, r\ sin\ \theta, z)\ r\ dz\ dr\ d\theta$$

What are spherical coordinates? Like cylindrical coordinates, spherical coordinates extend polar coordinates to three dimensions ($\mathbb{R}^3$). Spherical coordinates are related to the longitude and latitude coordinates used in navigation. They are represented by points of the form (ρ, θ, ϕ) where

$\rho =$ the distance from the origin to the point P $(\rho \geq 0)$

$\theta =$ the polar angle, as in polar coordinates $(0 \leq \theta \leq 2\pi)$

$\phi =$ the angle measured down from the positive z-axis to the ray from the origin through P $(0 \leq \phi \leq \pi)$



Triple Integrals in Spherical Coordinates If f is continuous on the bounded solid region D, then the triple integral of f over D is given by

$$\iiint_D f(x,y,z)\ dV = \iiint_{\bar{D}} f(\rho\ sin\ \phi\ cos\ \theta, \rho\ sin\ \phi\ sin\ \theta, \rho\ cos\ \phi)\ \rho^2\ sin\ \phi\ d\rho\ d\theta\ d\phi$$

where $\bar{D}$ is the region D expressed in spherical coordinates.

When to Use Cylindrical vs. Spherical Coordinates

It can be tricky when determining whether to use cylindrical or spherical coordinates to evaluate triple integrals. Either one may be used as both result in the same answer, but which is the most helpful? When integrating over a sphere - use spherical coordinates, when integrating over a cylinder - use cylindrical coordinates. In other instances, it may require some intuition or guess-and-check work to decide.

Conversion Formulas

Conversion Formulas for Polar Coordinates

$x = rcos\ \theta \qquad y = rsin\ \theta \qquad r = \sqrt{x^2+y^2} \qquad tan\ \theta = \frac{y}{x}$

Conversion Formulas for Cylindrical Coordinates

Cylindrical to Rectangular
(r, θ, z) to (x, y, z)
$x = rcos\ \theta \qquad y = rsin\ \theta \qquad z = z$

Rectangular to Cylindrical
(x, y, z) to (r, θ, z)
$r = \sqrt{x^2+y^2} \qquad tan\ \theta = \frac{y}{x} \qquad z = z$

Conversion Formulas for Spherical Coordinates

Spherical to Rectangular
(ρ, θ, ϕ) **to** (x, y, z)
$x = \rho\ sin\ \phi\ cos\ \theta$
$y = \rho\ sin\ \phi\ sin\ \theta$
$z = \rho\ cos\ \phi$

Rectangular to Spherical
(x, y, z) **to** (ρ, θ, ϕ)
$\rho = \sqrt{x^2+y^2+z^2}$
$tan\ \theta = \frac{y}{x}$
$\phi = cos^{-1}(\frac{z}{\sqrt{x^2+y^2+z^2}})$

Spherical to Cylindrical
(ρ, θ, ϕ) **to** (r, θ, z)
$r = \rho\ sin\ \phi$
$\theta = \theta$
$z = \rho\ cos\ \phi$

Cylindrical to Spherical
(r, θ, z) **to** (ρ, θ, ϕ)
$\rho = \sqrt{r^2+z^2}$
$\theta = \theta$
$\phi = cos^{-1}(\frac{z}{\sqrt{r^2+z^2}})$