Math 2450: Line and Surface Integrals

Line Integral

If $f(x, y, z)$ is defined on the smooth curve C with parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$, then the **line integral** of f over C is given by

$$\int_C f(x, y, z)dx = \lim_{\|\Delta s\| \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta s_k$$

provided that this limit exists. If C is a closed curve, we sometimes indicate the line integral of f around C by $\oint_C f ds$.

Written in terms of t, we have the line integral given by

$$\int_C f(x, y, z)ds = \int_a^b f(x(t), y(t), z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}dt$$

Note: Line integrals follow the rules of integrals, including the constant multiple rule and sum rule, which are helpful when finding line integrals of functions defined on piecewise smooth curves. It is also true that

$$\int_{-C} f ds = -\int_C f ds$$

where $-C$ denotes the curve C traversed in the opposite direction.

Line Integral of a Vector Field

Let $\mathbf{F}(x, y, z) = u(x, y, z)\mathbf{i} + v(x, y, z)\mathbf{j} + w(x, y, z)\mathbf{k}$ be a vector field, and let C be a piecewise smooth orientable curve with parametric representation

$$\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad \text{for } a \le t \le b$$

Using $d\mathbf{R} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$, we define the **line integral of F along C** by

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{R} &= \int_C (u\ dx + v\ dy + w\ dz) \\ &= \int_C \mathbf{F}[\mathbf{R}(t)] \cdot \mathbf{R}'(t)\ dt \\ &= \int_a^b [u[x(t), y(t), z(t)]\tfrac{dx}{dt} + v[x(t), y(t), z(t)]\tfrac{dy}{dt} + w[x(t), y(t), z(t)]\tfrac{dz}{dt}]\ dt\end{aligned}$$

Application of Line Integrals: Work

Let $\mathbf{F}$ be a continuous force field over a domain D. Then the work W performed as an object moves along a smooth curve C in D is given by the integral

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\ ds$$

where $\mathbf{T}$ is the unit tangent at each point on C.

Fundamental Theorem for Line Integrals

Let C be a piecewise smooth curve that is parameterized by the vector function $\mathbf{R}(t)$ for $a \le t \le b$, and let $\mathbf{F}(t)$ be a vector field that is continuous on C. If f is a scalar function such that $\mathbf{F} = \Delta f$, then

$$\int_C \mathbf{F} \cdot d\mathbf{R} = f(Q) - f(P)$$

where $Q = \mathbf{R}(b)$ and $P = \mathbf{R}(a)$ are the endpoints of C.

Conservative Vector Field

A vector field $\mathbf{F}$ is said to be **conservative** in a region D if $\mathbf{F} = \Delta f$ for some scalar function f in D. The function f is called a **scalar potential** of $\mathbf{F}$ in D. That is,

$$\mathbf{F} = \Delta f \quad \text{for } (x, y) \text{ on } D$$

Independence of Path

The line integral $\int_C \mathbf{F} \cdot d\mathbf{R}$ is **independent of path** in a region D if for any two points P and Q in D the line integral along every piecewise smooth curve in D from P to Q has the same value.

Equivalent Conditions for Path Independence

If $\mathbf{F}$ is a continuous vector field on the open connected set D, then the following three conditions are either all true or all false:

(**i**) $\mathbf{F}$ is conservative on D; that is, $\mathbf{F} = \Delta f$ for some function f defined on D.

(**ii**) $\oint_C \mathbf{F} \cdot d\mathbf{R} = 0$ for every piecewise smooth closed curve C in D.

(**iii**) $\int_C \mathbf{F} \cdot d\mathbf{R}$ is independent of path within D.

(**iv**) If $\mathbf{F}$ is a continuously differentiable vector field with *curl* $\mathbf{F} = 0$

Surface Integral

Let S be a surface defined by $z = f(x, y)$ and R be its projection on the xy-plane. If f, f_x, f_y are continuous in R and g is a continuous function of three variables on S, then the **surface integral** of g over S is

$$\iint_S \; g(x, y, z) \; dS = \iint_R \; g(x, y, f(x, y))\sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \; dA$$

Note: If the surface S is defined parametrically by $\mathbf{R}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$ over a region D in the uv-plane, we saw previously in the course that the **surface area** of S is given by

$$\iint_D \|\mathbf{R_u} \times \mathbf{R_v}\| \, dudv$$

Likewise, if f is continuous on D, then we may express the surface integral of f over D as below

$$\iint_S f(x, y, z) \; dS = \iint_D f(\mathbf{R}) \, \|\mathbf{R_u} \times \mathbf{R_v}\| \; dudv$$

Applications of Surface Integrals

Surface Area If S is a piecewise smooth surface, its area is given by

$$A = \iint_S \; dS$$

Flux Integral Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on the surface S, which is oriented by the unit normal field $\mathbf{N}$. Then the **flux** of $\mathbf{F}$ across S is given by the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{N} \; dS = \iint_R \mathbf{F}(x, y, f(x, y)) \cdot \langle -f_x, -f_y, 1 \rangle \; dA$$

where the surface S is described by $z = f(x, y)$ with an upward unit normal vector.