Math 2450: Applications of Multiple Integrals

We have explored how to properly set up multiple integrals over a specified domain, evaluate the integrals, and change the order of integration. Now we can apply those skills to calculate averages, areas between curves, surface areas, and volumes enclosed by a surface.

Calculating Area of a Region	**Calculating Volume of a Solid**
The area of the region D in the xy-plane is given by $A = \iint_D \, dA$ **Note:** This reduces to the single integral case from Calc I after we evaluate the first integral. Using a double integral gives us the ability to change order of integration if necessary.	The volume of the solid region E is given by $V = \iiint_E \, dV$ **Note:** This reduces to the double integral formula (diagonal) after we evaluate the first integral.
Calculating Volume of a Bounded Solid	**Calculating the Total Mass of a Solid**
When $f(x, y) \geq 0$ and continuous on a region D the volume of a solid bounded above by $z = f(x, y)$and below by D in the xy-plane is given by $V = \iint_D f(x, y) \, dA$	$f(x, y, z)$ is the density of a solid object D at a given point (x, y, z). The the total mass of the solid is given by $\iiint_D f(x, y, z) \, dV$

Average Values

The average value of the continuous function f over R is given by:

One Variable	Two Variables	Three Variables
$\frac{1}{\textit{length of segment } R} \int_R f(x)dx$	$\frac{1}{\textit{area of region } R} \iint_R f(x, y)dA$	$\frac{1}{\textit{volume of solid } R} \iiint_R f(x, y, z)dV$

Surface Area

Assume that the function $f(x, y)$ has continuous partial derivatives f_x and f_y in a region R of the xy-plane. Then the portion of the surface $z = f(x, y)$ that lies over R has surface area

$$S = \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} \, dA$$

We also have the case when the surface $y = f(x, z)$ is projected onto the region Q in the xz-plane. The formula for surface area becomes

$$S = \iint_Q \sqrt{[f_x(x, z)]^2 + [f_z(x, z)]^2 + 1} \, dx \, dz$$

And similarly, for projection onto the region T in the yz-plane would have surface area

$$S = \iint_T \sqrt{[f_y(y, z)]^2 + [f_z(y, z)]^2 + 1} \, dy \, dz$$

Parametric Definition of Surface Area

Let S be a surface defined parametrically by $\mathbf{R}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$ on the region D in the uv-plane, and assume that S is smooth in the sense that $\mathbf{R}_u$ and $\mathbf{R}_v$ are continuous with $\mathbf{R}_u \times \mathbf{R}_v \neq 0$ on D. Then the surface area, S, is defined by

$$S = \iint_D \|\mathbf{R}_u \times \mathbf{R}_v\| \, du \, dv$$

The quantity $\|\mathbf{R}_u \times \mathbf{R}_v\|$ is called the fundamental cross product.