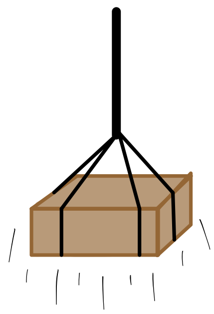
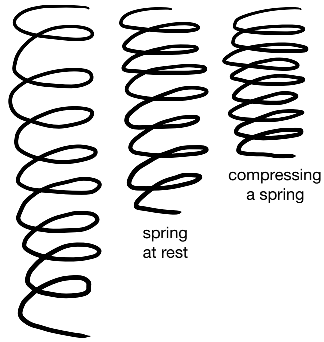
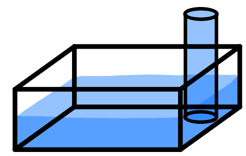
Math 1452: Work Word Problems

What is work? In general, work is a measure of how much effort it takes to apply some force over a certain distance. Below are some illustrations of work:







How do we calculate work? You may be familiar with the following formula for simple work:

$$\text{Work} = \text{Force} \times \text{Distance}$$

In the formula above, we calculate work by multiplying a constant force times the distance over which we apply the force. This work is *simple* because the force is constant over the entire distance. If the force changes depending on the distance, we can no longer use simple multiplication to calculate work. Instead, we treat distance as a variable, write the force as a function depending on distance, and add up force $\times$ distance at each point using an integral. Remember from Calculus I that an integral is just a summation where the summand changes. Below are formulas for the types of work illustrated above, all of which depend on distance:

Type of Work	Formula for Force
Lifting Objects	F = weight density $\times$ distance
Using Springs	F = spring constant $\times$ distance
Pumping Liquid	F = weight density $\times$ distance $\times$ area of a slice

Notice that two of the above equations depend on weight density. In some questions, the weight density will be given to you and in others, it is a constant for you to remember.
The weight density of water is 62.4 pounds per cubic foot.

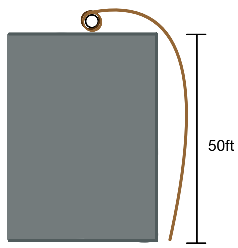
How do I set up my integral? The integrand will be one of the formulas for force listed above using either x or y variables. Since every force formula depends on distance, the distance will determine the variable. If the object is moving *vertically* we use x and if the object is moving *horizontally* we use y. Regarding the integral bounds, for most of these questions, the lower bound of the integral will be 0 and the upper bound of the integral will be the total distance that the object moves. For each type of work, the upper bound is as follows:

Type of Work	Upper Bound
Lifting Objects	Distance lifted
Using Springs	Distance stretched or compressed
Pumping Liquid	Height of liquid

What unit do I use for work? Work is measured in joules, so the answer for each work question will be the evaluation of our integral followed by “joules”.

Example 1. A cable is 50 meters long and has a density of 3 kg/m. If the cable is hanging over the side of the building, then what is the work done in pulling the cable up?

Let's start by pulling out all the numbers from the question and drawing a sketch of the scenario:



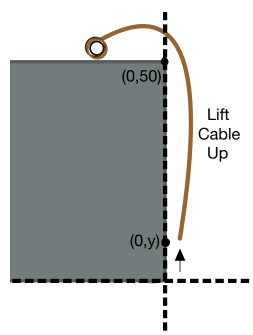
Quantity	Description
50 meters	length of cable
3 kg/m	weight density of cable

1. **Identifying the integrand:** We need to know the **weight density** and the **distance**.

   (a) **Weight Density:** As we can see from the known information, the weight density is 3.

   (b) **Distance:** Since the cable is moving *vertically*, we will use y as the variable to represent the height of the end of the cable. The distance we have pulled the cable is the same as the height of the cable, represented by y. Therefore the remaining distance for the cable to be pulled is represented by $50 - y$. Therefore, the force is represented by the equation

$$F = 3 \times (50 - y).$$



2. **Identifying the bounds:** This means that the value of y starts at 0 when the cable is on the ground and ends at 50 when we have pulled the cable 50 feet up the side of the building. Therefore our lower bound is 0 and our upper bound is 50.

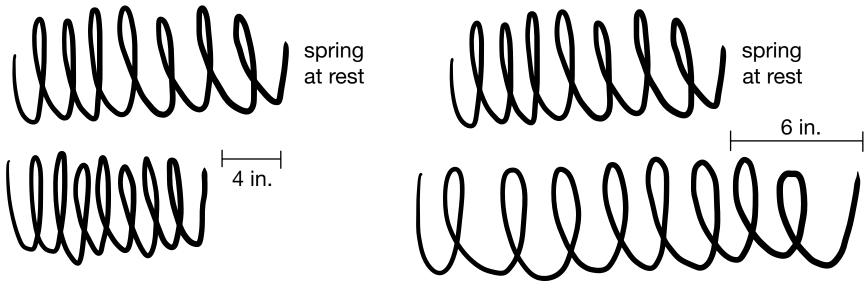
Now that we have identified our integrand and our bounds, we set up the integral as

$$\begin{aligned}
&\int_0^{50} 3 \times (50 - y) dy \\
&= 3 \int_0^{50} 50 - y dy \\
&= 3 \left(50y - \frac{1}{2} y^2 \Big|_0^{50} \right) \\
&= 3 \left(\left(2500 - \frac{1}{2} \cdot 2500 \right) - \left(50 \cdot 0 - \frac{1}{2} \cdot 0 \right) \right) \\
&= 3 (1250) \\
&= 3750 \text{ joules}
\end{aligned}$$

Example 2. A force of 3 lbs is required to hold a spring that has been compressed 4 inches from its natural length. Find the work done in stretching the spring 6 inches from its natural length.

Let's start by pulling out all the numbers from the question and drawing a sketch of the scenario:

Quantity	Description
3 lbs	force required to compress 4 inches
6 inches	length stretched



1. **Identifying the integrand:** We need to know the **spring constant** and the **distance**.

   (a) **Spring Constant:** For all questions with springs, there will be a sentence in the question that gives the force required to stretch or compress the spring a certain length. This sentence will allow us to solve for the spring constant k. In this case, a force of 3 lbs is required to compress 4 inches, so we solve:

   $$\text{F} = k \times \text{distance} \quad \longrightarrow \quad 3 = k \times 4 \quad \longrightarrow \quad \frac{3}{4} = k$$

   (b) **Distance:** Since the spring is being stretched and compressed *horizontally*, we will use x as our variable to represent the distance that the spring has been stretched.

2. **Identifying the bounds:** Since the spring will compress 6 inches, our lower bound is 0, representing our spring starting at rest, and our upper bound is 6, representing our spring compressing 6 inches.

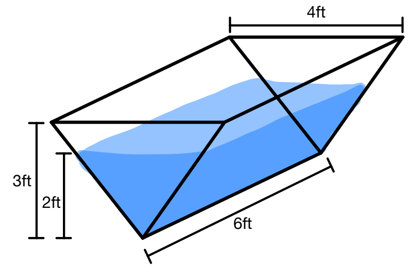
Now that we have identified our integrand and our bounds, we set up the integral as

$$\begin{aligned}
&\int_0^6 \frac{3}{4} x dx \\
&= \frac{3}{8}x^2 \Big|_0^6 \\
&= \left(\frac{3}{8} \cdot 6^2 - \frac{3}{8} \cdot 0\right) \\
&= \frac{108}{8} \\
&= \frac{27}{2} \text{ joules}
\end{aligned}$$

Example 3. A tank is shaped like a trough with a triangular base. It is 3 feet high, 6 feet long, and 4 feet wide. It contains water to a depth of 2 feet. Find the work required to pump all the water to the top of the tank.

Let's start by pulling out all the numbers from the question and drawing a sketch of the scenario:

Quantity	Description
3 feet	height of tank
4 feet	width of tank
6 feet	length of tank
2 feet	water level

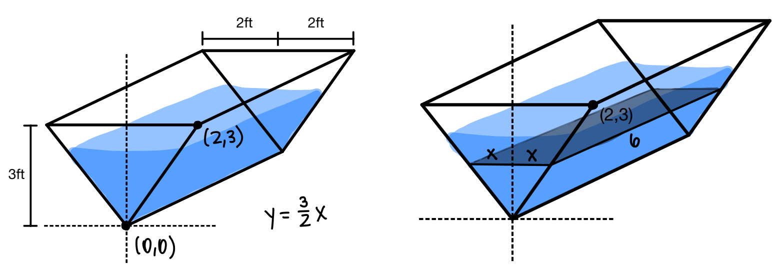


1. **Identifying the integrand:** We need to know the **weight density**, **distance**, and the **area of a slice**.

   (a) **Weight Density:** Since our liquid is water, the weight density is 62.4.

   (b) **Distance:** Since the water is being pumped our of the tank it is moving *vertically* out of the tank and we will use y as our variable to represent the current water level, which decreases as the water is pumped out. The distance we pump the water increases as the water level decreases and is represented by the quantity $3 - y$.

   (c) **Area of a Slice:** To find the area of a slice, we put a coordinate plane on our sketch with the origin at the base of the triangle. We shade a slice of the water as a rectangle and find the area of the slice. The length of the rectangle is 6, but the width changes depending on the water level. To find the width of the rectangle, we need to solve for $2x$. However, we have already used y in the distance, so we need to write the width, $2x$, in terms of its height y. We identify the point $(2, 3)$ on our triangle and find the equation $y = \frac{3}{2}x$.



$$y = \frac{3}{2}x \quad \longrightarrow \quad \frac{2}{3}y = x \quad \longrightarrow \quad \frac{4}{3}y = 2x$$

   Therefore, the area of a slice is $6 \cdot \frac{4}{3}y = 8y$.

2. **Identifying the bounds:** Since the water level is 2 feet, our lower bound is 0 and our upper bound is 2.

Now that we have identified our integrand and our bounds, we set up the integral as

$$\int_0^2 (62.4 \times (3 - y) \times 8y)\, dy = 449.2 \int_0^2 3y - y^2 dy.$$

For this final one, see if you can integrate on your own and obtain 1497.33 joules.