Math 1452: Trig Substitution

In Calculus II, it is important that we remember all of the antiderivative rules we learned in Calculus I. Let's recall three of the antiderivative rules for the inverse trig functions:

$$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C \qquad \int \frac{1}{1+x^2}dx = \tan^{-1} x + C \qquad \int \frac{1}{|\,x\,|\sqrt{x^2-1}}dx = \sec^{-1} x + C$$

What do these integrals have in common? One thing each of these integrals has in common is an x^2 term in the denominator that is added to or subtracted from 1. These terms in the denominator look similar to the square roots below:

$$\sqrt{1-x^2} \qquad \sqrt{1+x^2} \qquad \sqrt{x^2-1}$$

A common type of integral that we will see in Calculus II are integrals with this square root in the integrand, or a similar square root where the 1 is replaced with another square value, like the ones below:

$$\sqrt{a^2-x^2} \qquad \sqrt{a^2+x^2} \qquad \sqrt{x^2-a^2}$$

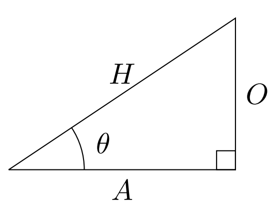
What strategy do I use to evaluate these integrals? If our integrand has a square root that looks like this, we will use a strategy called *trig substitution*, in which we use u-substitution combined with the Pythagorean identities

$$\sin^2(x) + \cos^2(x) = 1 \quad \text{and} \quad \tan^2(x) + 1 = \sec^2(x)$$

Instead of using a u for our substitution, we will use θ. We show this substitution process for the three types of square roots highlighted above:

$$\begin{array}{rrr}
\sqrt{a^2-x^2} & \sqrt{a^2+x^2} & \sqrt{x^2-a^2} \\
x = a\sin(\theta) & x = a\tan(\theta) & x = a\sec(\theta) \\
\downarrow & \downarrow & \downarrow \\
\sqrt{a^2 - a^2\sin^2(\theta)} & \sqrt{a^2 + a^2\tan^2(\theta)} & \sqrt{a^2\sec^2(\theta) - a^2} \\
= \sqrt{a^2(1-\sin^2(\theta))} & = \sqrt{a^2(1+\tan^2(\theta))} & = \sqrt{a^2\left(\sec^2(\theta)-1\right)} \\
= \sqrt{a^2\cos^2(\theta)} & = \sqrt{a^2\sec^2(\theta)} & = \sqrt{a^2\tan^2(\theta)} \\
= a\cos(\theta) & = a\sec(\theta) & = a\tan(\theta)
\end{array}$$

To solve for θ, we create a *reference triangle* and use SOHCAHTOA.



$$\sin(\theta) = \frac{O}{H}, \quad \tan(\theta) = \frac{O}{A}, \quad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{H}{A}$$

Example 1. Evaluate the integral

$$\int \frac{5}{x^2\sqrt{9+x^2}}dx$$

The square root in the integrand is in the form matching $\tan^{-1}(x)$, so we set $x = 3\tan(\theta)$.

$$\begin{aligned} x &= 3\tan(\theta) \\ \frac{dx}{d\theta} &= 3\sec^2(\theta) \\ dx &= 3\sec^2(\theta)d\theta \end{aligned} \qquad \begin{aligned} x^2 &= 9\tan^2(\theta) \\ \sqrt{9+x^2} &= \sqrt{9+9\tan^2(\theta)} \\ &= \sqrt{9(1+\tan^2(\theta))} \\ &= \sqrt{9\sec^2(\theta))} \\ &= 3\sec(\theta) \end{aligned}$$

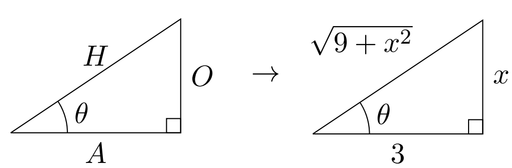
We use this substitution like we did in Calculus I.

$$\begin{aligned} \int \frac{5}{x^2\sqrt{9+x^2}}dx &= \int \frac{5}{9\tan^2(\theta)\cdot 3\sec(\theta)} \cdot 3\sec^2(\theta)d\theta \\ &= \int \frac{5\sec(\theta)}{9\tan^2(\theta)}d\theta \\ &= \frac{5}{9}\int \sec(\theta)\cdot\frac{1}{\tan^2(\theta)}d\theta \\ &= \frac{5}{9}\int \frac{1}{\cos(\theta)}\cdot\frac{\cos^2(\theta)}{\sin^2(\theta)}d\theta \\ &= \frac{5}{9}\int \frac{\cos(\theta)}{\sin^2(\theta)}d\theta \end{aligned}$$

At this point, we finish the integral by using the substitution $u = \sin(\theta)$ and $d\theta = \frac{du}{\cos(\theta)}$.

$$\begin{aligned} &\frac{5}{9}\int \frac{\cos(\theta)}{\sin^2(\theta)}d\theta \\ =&\frac{5}{9}\int \frac{\cos(\theta)}{u^2}\frac{du}{\cos(\theta)} \\ =&\frac{5}{9}\int \frac{1}{u^2}du \\ =&\frac{5}{9}\int u^{-2}du \\ =&\frac{5}{9}\cdot\frac{-1}{u}+C \\ =&\frac{-5}{9\sin(\theta)}+C \end{aligned}$$

Now, we draw a reference triangle to rewrite the answer in terms of x with the equality $\frac{x}{3} = \tan(\theta)$.



Therefore $\sin(\theta) = \frac{x}{\sqrt{9+x^2}}$ and our integral evaluates to $\frac{-5\sqrt{9+x^2}}{9x} + C$.

Example 2. Evaluate the integral

$$\int \frac{\sqrt{x^2-4}}{x}dx$$

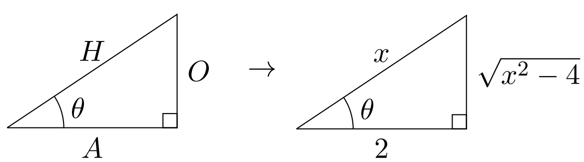
The square root in the integrand is in the form matching $\sec^{-1}(x)$, so we set $x = 2\sec(\theta)$.

$$\begin{aligned} x &= 2\sec(\theta) \\ \frac{dx}{d\theta} &= 2\sec(\theta)\tan(\theta) \\ dx &= 2\sec(\theta)\tan(\theta)d\theta \end{aligned} \qquad \begin{aligned} x^2 &= 4\sec^2(\theta) \\ \sqrt{x^2-4} &= \sqrt{4\sec^2(\theta)-4} \\ &= \sqrt{4(\sec^2(\theta)-1)} \\ &= \sqrt{4\tan^2(\theta))} \\ &= 2\tan(\theta) \end{aligned}$$

We use this substitution like we did in Calculus I.

$$\begin{aligned} \int \frac{\sqrt{x^2-4}}{x}dx &= \int \frac{2\tan(\theta)}{2\sec(\theta)} \cdot 2\sec(\theta)\tan(\theta)d\theta \\ &= \int 2\tan^2(\theta)d\theta \\ &= 2\int \tan^2(\theta)d\theta \\ &= 2\int \sec^2(\theta) - 1d\theta \\ &= 2\int \sec^2(\theta) - 2\int 1d\theta \\ &= 2\tan(\theta) - 2\theta + C \end{aligned}$$

At this point, we draw a reference triangle to rewrite the answer in terms of x with the information that $\frac{x}{2} = \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{H}{A}$.



Therefore $\theta = \sec^{-1}\left(\frac{x}{2}\right)$ and $\tan(\theta) = \frac{\sqrt{x^2-4}}{2}$, so our integral evaluates to

$$2 \cdot \frac{\sqrt{x^2-4}}{2} + 2\cdot\sec^{-1}\left(\frac{x}{2}\right) + C = \sqrt{x^2-4} + 2\sec^{-1}\left(\frac{x}{2}\right) + C.$$

Example 3. Evaluate the integral

$$\int \frac{1}{x^2\sqrt{4-x^2}}dx$$

Try this example out on your own. For your answer, you should get $\frac{-\sqrt{4-x^2}}{4x} + C$.