Math 1452: Integrating Powers of Trig Functions

A common integral type has the integrand as a power of complementary trig functions, such as the examples below:

$$\int \sin^2(x)\cos^2(x)dx, \quad \int \sec^4(x)\tan^3(x)dx, \quad \int \csc^3(x)\cot(x)dx$$

Each of these integrals can be solved using the integral techniques from Calculus I, using u-substitution and trig identities. However, there are some tricks to be able to tell which substitutions and identities we will use. Overall, the goal of rewriting the integrand is to change the integral to easily use u-substitution. Therefore, our goal is to factor out one of the following expressions:

- $\sin(x)$
- $\cos(x)$
- $\sec(x)\tan(x)$
- $\sec^2(x)$
- $\csc(x)\cot(x)$
- $\csc^2(x)$

with the remaining trig functions in our integrand having an even power. We will look at each pair of complementary trig functions separately.

Powers of sin(x) and cos(x). These simplifications will depend on if the powers of $\sin(x)$ and $\cos(x)$ are even or odd and will rely on the trig identity $\sin^2(x) + \cos^2(x) = 1$.

i. If $\cos(x)$ has odd power, then factor out one $\cos(x)$ and change the remaining even powers to $\sin(x)$ using the identity $\cos^2(x) = 1 - \sin^2(x)$.

ii. If $\sin(x)$ has odd power, then factor out one $\sin(x)$ and change the remaining even powers to $\cos(x)$ using the identity $\sin^2(x) = 1 - \cos^2(x)$.

iii. If both of the powers are even, we will use the trig identities

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \quad \text{and} \quad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

to change the integrand to be in terms of powers of $\cos(x)$.

Powers of sec(x) and tan(x). These simplifications will depend on if the powers of $\sec(x)$ and $\tan(x)$ are even or odd and will rely on the trig identity $\tan^2(x) + 1 = \sec^2(x)$.

a. If $\sec(x)$ has even power, then factor out $\sec^2(x)$ and change the remaining even powers to $\tan(x)$ using the identity $\sec^2(x) = \tan^2(x) + 1$.

b. If $\tan(x)$ has odd power, then factor out $\sec(x)\tan(x)$ and change the remaining even powers to $\sec(x)$ using the identity $\tan^2(x) = \sec^2(x) - 1$.

c. If $\tan(x)$ has even power, then we will change the $\tan(x)$ terms to $\sec(x)$ using the identity $\tan^2(x) + 1 = \sec^2(x)$ and then use either antidifferentition or integration by parts.

Powers of cot(x) and csc(x). These simplifications work in the exact same manner as the box above after replacing $\sec(x)$ with $\csc(x)$ and replacing $\tan(x)$ with $\cot(x)$, making sure to now use the trig identity $1 + \cot^2(x) = \csc^2(x)$.

Example 1. Evaluate the integral $\int \sin^3(x)\cos^4(x)dx$.
In this example, the power of $\sin(x)$ is 3, which is odd and the power of $\cos(x)$ is 4, which is even. So we are in case ii. and we factor out $\sin(x)$ and change the remaining $\sin^2(x)$ to $1-\cos^2(x)$.

$$\begin{aligned}\int \sin^3(x)\cos^4(x)dx &= \int \sin(x)\sin^2(x)\cos^4(x)dx \\ &= \int \sin(x)\left(1-\cos^2(x)\right)\cos^4(x)dx \\ &= \int \sin(x)\left(\cos^4(x)-\cos^6(x)\right)dx \\ &= \int \sin(x)\cos^4(x)dx - \int \sin(x)\cos^6(x)dx\end{aligned}$$

From this point, we use u-substitution with $u=\cos(x)$ and $dx = \frac{du}{\sin(x)}$.

Example 2. Evaluate the integral $\int \sec^4(x)\tan^3(x)dx$.
In this example, the power of $\sec(x)$ is 4, which is even and the power of $\tan(x)$ is 3, which is odd. This fits case a. and b., so we can either factor out $\sec(x)\tan(x)$ or factor out $\sec^2(x)$. We will choose to factor out $\sec^2(x)$:

$$\begin{aligned}\int \sec^4(x)\tan^3(x)dx &= \int \sec^2(x)\tan^3(x)\sec^2(x)dx \\ &= \int (\tan^2(x)+1)\tan^3(x)\sec^2(x)dx \\ &= \int (\tan^5(x)+\tan^3(x))\sec^2(x)dx\end{aligned}$$

From this point, we use u-substitution with $u=\tan(x)$ and $dx = \frac{du}{\sec^2(x)}$.

Example 3. Evaluate the integral $\int \csc^3(x)\cot(x)dx$.
In this example, the power of $\cot(x)$ is 1, which is odd and the power of $\csc(x)$ is 3, which is also odd. In this case, we factor out $\csc(x)\cot(x)$ and then use substitution since there are no remaining powers of $\cot(x)$.

$$\int \csc^3(x)\cot(x)dx = \int \csc^2(x)\csc(x)\cot(x)dx$$

From this point, we use u-substitution with $u=\csc(x)$ and $dx = \frac{du}{\csc(x)\cot(x)}$.

There are just some examples of how to apply these integration techniques. The best way to improve your skills at integration is to continue practicing. More examples using these integration techniques can be found here: https://tutorial.math.lamar.edu/Classes/CalcII/IntegralsWithTrig.aspx