Math 1452: Infinite Series Tests

In this worksheet, we will look at different methods to determine converge for infinite series. First, we will look at the four known types of series. Next, we will describe six tests to analyze positive term series. Finally, we will describe four tests to analyze series with positive *and* negative terms.

Known Types of Series:

p-Series	**q-Log Series**
The general form of this series is $\sum_{k=1}^{\infty} \frac{1}{k^p}$ for some power p. • If $p > 1$, then the series will converge. • If $p \leq 1$, then the series will diverge.	The general form of this series is $\sum_{k=1}^{\infty} \frac{\ln(k)}{k^q}$ for some power q. • If $q > 1$, then the series will converge. • If $q \leq 1$, then the series will diverge.
Geometric Series	**Telescoping Series**
The general form of this series is $\sum_{k=1}^{\infty} a_k$ where $a_k = ar^k$ for some real numbers a and r. • If $r < 1$, then the series will converge to $\frac{a}{1-r}$. • If $r \geq 1$, then the series will diverge.	The general form of this series is $\sum_{k=1}^{\infty} a_k$ where a_k is a difference of fractions. We expand the summation and cancel common terms to find a general formula for the n^{th} parital sum S_n. • If $\lim_{n\to\infty} S_n < \infty$, then the series converges. • If $\lim_{n\to\infty} S_n = \infty$, then the series diverges.

Tests for Positive Term Series:

Divergence Test	**Integral Test**
Consider a series $\sum_{k=1}^{\infty} a_k$. • If $\lim_{n\to\infty} a_k = 0$, then the test is inconclusive. • If $\lim_{n\to\infty} a_k \neq 0$, then the series diverges.	Consider a series $\sum_{k=1}^{\infty} a_k$. • If $\int_1^{\infty} a(x)dx$ converges, the series converges. • If $\int_1^{\infty} a(x)dx$ diverges, the series diverges.
Ratio Test	**Root Test**
Given $\sum_{k=1}^{\infty} a_k$, find a general form for a_{k+1}. • If $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$, then $\sum_{k=1}^{\infty} a_k$ converges. • If $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges. • If $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$, then the test is inconclusive.	Given $\sum_{k=1}^{\infty} a_k$, find a general form for $\sqrt[k]{a_k}$. • If $\lim_{k\to\infty} \sqrt[k]{a_k} < 1$, then $\sum_{k=1}^{\infty} a_k$ converges. • If $\lim_{k\to\infty} \sqrt[k]{a_k} > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges. If $\lim_{k\to\infty} \sqrt[k]{a_k} = 1$, then the test is inconclusive.
Direct Comparison Test	**Limit Comparison Test**
Given $\sum_{k=1}^{\infty} a_k$, choose a similar series $\sum_{k=1}^{\infty} b_k$. • If $a_k < b_k$ and $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges. • If $a_k > b_k$ and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.	Given $\sum_{k=1}^{\infty} a_k$, choose a similar series $\sum_{k=1}^{\infty} b_k$. • If $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges. • If $\lim_{k\to\infty} \frac{a_k}{b_k} = \infty$ and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges. • If $\lim_{k\to\infty} \frac{a_k}{b_k} = L$ and $0 < L < \infty$, then $\sum_{k=1}^{\infty} a_k$ copies $\sum_{k=1}^{\infty} b_k$.

Tests for a Series with Positive and Negative Terms:

For a series $\sum_{k=1}^{\infty} a_k$ with positive and negative terms, we will check for two types of convergence: absolute and conditional. To check for absolute convergence, we look at the series $\sum_{k=1}^{\infty} |a_k|$ and use any of the tests above or one of the following:

Generalized Ratio Test

Given $\sum_{k=1}^{\infty} a_k$, determine a general form for a_{k+1}.

- If $\lim\limits_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1$, then $\sum_{k=1}^{\infty} a_k$ converges absolutely.
- If $\lim\limits_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| > 1$, then $\sum_{k=1}^{\infty} |a_k|$ diverges.
- If $\lim\limits_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = 1$, then the test is inconclusive.

Generalized Root Test

Given $\sum_{k=1}^{\infty} a_k$, determine a general form for $\sqrt[k]{a_k}$.

- If $\lim\limits_{k\to\infty} \sqrt[k]{|a_k|} < 1$, then $\sum_{k=1}^{\infty} a_k$ converges absolutely.
- If $\lim\limits_{k\to\infty} \sqrt[k]{|a_k|} > 1$, then $\sum_{k=1}^{\infty} |a_k|$ diverges.
- If $\lim\limits_{k\to\infty} \sqrt[k]{|a_k|} = 1$, then the test is inconclusive.

What if $\sum_{k=1}^{\infty} |a_k|$ diverges? In this case, we switch to checking for conditional convergence, which uses one final test described below:

Alternating Series Test

Given $\sum_{k=1}^{\infty} (-1)^k a_k$, find a general form for a_{k+1}.

If $\lim\limits_{k\to\infty} a_k = 0$ and $a_{k+1} < a_k$, then $\sum_{k=1}^{\infty} a_k$ converges conditionally.