Math 1452: Sequences vs. Series

What is a sequence? A *sequence* is a function from the positive integers to the real numbers, written with function notation as $a(n)$. Consider the example $a(n) = \frac{1}{n}$.

$$\begin{array}{cccccccccccc} n & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\ \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ a(n) & & & & & & & & & & & \end{array}$$

Typically, we will omit the parenthesis in $a(n)$ and instead write the *general term* a_n, listing the sequence as $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots\}$, or in the same example as above, listing the sequence as

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} =$$

In this way, we represent a_n as a list of outputs from the function, starting with input 1 and increasing one integer at a time.

What is a series? A *series* is a mathematical summation, written as $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$. The term a_n is a placeholder for some algebraic expression involving n, and this is called the *general term* of the series. Again, let's consider the example $a_n = \frac{1}{n}$ and look at the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\sum_{n=1}^{\infty} \frac{1}{n} =$$

What do we do with sequences and series? One of the main objectives of sequences and series is determining their *convergence*.

- Sequence convergence occurs when

- Series convergence occurs when

What new tools do we have to determine convergence? One strategy we will use to determine convergence is the *tower of power*, which compares common expressions and how their growth relates to one another.

SLOWER GROWTH	FASTER GROWTH

With the tower of power, expressions a_n in which the denominator is higher on the tower of power than the numerator have $\lim_{n\to\infty} a_n = 0$. In these cases, the denominator has faster growth than the numerator, and when the value of n is large enough, the limit will evaluate to $\frac{1}{\infty} = 0$. Some examples of this are below:

$$\lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} = \qquad \lim_{n\to\infty} \frac{n}{e^n} = \qquad \lim_{n\to\infty} \frac{\ln(n)}{n^p} = \qquad \lim_{n\to\infty} \frac{n!}{n^n} =$$

Let's look at some examples determining convergence of sequences and series.

Example 1. Determine if the sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n = \frac{1}{2^n}$ converges or diverges.

To determine if this sequence converges, we evaluate

$$\lim_{n\to\infty} a_n =$$

Example 2. Determine if the series $\sum_{n=1}^{\infty} a_n$ with $a_n = \frac{1}{2^n}$ converges or diverges.

To determine if this series converges, we write out the first few partial sums S_k for $k = 1, 2, 3 \dots$.

$$S_1 =$$

$$S_2 =$$

$$S_3 =$$

$$S_4 =$$

To determine if the series converges, we evaluate the limit

$$\lim_{k\to\infty} S_k =$$

Example 3. Determine if the sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n = \frac{\ln(n)}{n^2}$ converges or diverges.

To determine if this sequence converges, we evaluate

$$\lim_{n\to\infty} a_n =$$

Example 4. Determine if the series $\sum_{n=1}^{\infty} a_n$ with $a_n = \frac{\ln(n)}{n^2}$ converges or diverges.

To determine if this series converges, we write out the first few partial sums S_k for $k = 1, 2, 3 \dots$.

$$S_1 =$$

$$S_2 =$$

$$S_3 =$$

$$S_4 =$$

At this point, there is not a clear pattern to write a general formula S_k. Therefore we will need to use some additional strategies described in Sections 8.2-8.6 of the textbook, as well as in the "Infinite Series Tests" worksheet found in the same place you accessed this worksheet.