Math 1452: Sequences vs. Series

What is a sequence? A *sequence* is a function from the positive integers to the real numbers, written with function notation as $a(n)$, with n as the independent variable. Consider the example $a(n) = \frac{1}{n}$. We evaluate this function on the first few positive integers below.

$$\begin{array}{ccccccccccc} n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ a(n) & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \dots \end{array}$$

Typically, we will omit the parenthesis in $a(n)$ and instead write the *general term* a_n, listing the sequence as $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \dots\}$, or in the same example as above, listing the sequence as

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \dots\right\}$$

In this way, we represent a_n as a list of outputs from the function, starting with input 1 and increasing one integer at a time.

What is a series? A *series* is a mathematical summation. We have seen these summations before in Calculus I to define an integral, so you may remember the notation for an infinite series as $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$. The term a_n is a placeholder for some algebraic expression involving n, and this is called the *general term* of the series. Again, let's consider the example $a_n = \frac{1}{n}$ and look at the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \dots$$

What do we do with sequences and series? One of the main objectives of sequences and series is determining their *convergence*.

- Sequence convergence occurs when $\lim_{n\to\infty} a_n$ exists and is finite. Evaluating this limit will use the same limit strategies that we learned in Calculus I.
- Series convergence occurs when $\sum_{n=1}^{\infty} a_n < \infty$. To evaluate this summation, we look at the n^{th} *partial sum* $S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$ and evaluate $\lim_{n\to\infty} S_n$. If the limit exists, then that is also the value the infinite series converges to.

What new tools do we have to determine convergence? One strategy we will use to determine convergence is the *tower of power*, which compares common expressions and how their growth relates to one another. In the box below, $p > 1$ is some power and $c > 1$ is some constant.

SLOWER GROWTH $\ln(n) \to \sqrt{n} \to n \to n^p \to c^n \to n! \to n^n$ FASTER GROWTH

With the tower of power, expressions a_n in which the denominator is higher on the tower of power than the numerator have $\lim_{n\to\infty} a_n = 0$. In these cases, the denominator has faster growth than the numerator, and when the value of n is large enough, the limit will evaluate to $\frac{1}{\infty} = 0$. Some examples of this are below:

$$\lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} = 0 \qquad \lim_{n\to\infty} \frac{n}{e^n} = 0 \qquad \lim_{n\to\infty} \frac{\ln(n)}{n^p} = 0 \qquad \lim_{n\to\infty} \frac{n!}{n^n} = 0$$

Let's look at some examples determining convergence of sequences and series.

Example 1. Determine if the sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n = \frac{1}{2^n}$ converges or diverges.

To determine if this sequence converges, we evaluate

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = \frac{1}{2^\infty} = 0.$$

In the expression $\frac{1}{2^n}$, the denominator grows faster than the numerator and therefore the limit evaluates to $\frac{1}{\infty} = 0$. Since this limit exists and is finite, the sequence converges to 0.

Example 2. Determine if the series $\sum_{n=1}^{\infty} a_n$ with $a_n = \frac{1}{2^n}$ converges or diverges.

To determine if this series converges, we write out the first few partial sums S_k for $k = 1, 2, 3 \ldots$.

$$\begin{aligned}
S_1 &= \frac{1}{2} \\
S_2 &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\
S_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \\
S_4 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}
\end{aligned}$$

From this pattern, we can see the general formula $S_k = \frac{2^k-1}{2^k}$. To determine if the series converges, we evaluate the limit

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{2^k-1}{2^k} = \lim_{k\to\infty} \frac{2^k}{2^k} - \frac{1}{2^k} = \lim_{k\to\infty} 1 - \lim_{k\to\infty} \frac{1}{2^k} = 1 - 0 = 1$$

Again, in the expression $\frac{1}{2^k}$, the denominator grows faster than the numerator and therefore the limit $\lim\limits_{k\to\infty} \frac{1}{2^k}$ evaluates to $\frac{1}{\infty} = 0$. Since this limit exists and is finite, the series converges.

Example 3. Determine if the sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n = \frac{\ln(n)}{n^2}$ converges or diverges.

To determine if this sequence converges, we evaluate

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln(n)}{n^2} = 0.$$

In the expression $\frac{\ln(n)}{n^2}$, the denominator grows faster than the numerator and therefore the limit evaluates to $\frac{1}{\infty} = 0$. Since this limit exists and is finite, the sequence converges to 0.

Example 4. Determine if the series $\sum_{n=1}^{\infty} a_n$ with $a_n = \frac{\ln(n)}{n^2}$ converges or diverges.

To determine if this series converges, we write out the first few partial sums S_k for $k = 1, 2, 3 \ldots$.

$$\begin{aligned}
S_1 &= \frac{\ln(1)}{1^2} = \frac{0}{1} = 0 \\
S_2 &= 0 + \frac{\ln(2)}{4} = \frac{\ln(2)}{4} \\
S_3 &= 0 + \frac{\ln(2)}{4} + \frac{\ln(3)}{9} = \frac{9\ln(2) + 4\ln(3)}{36} \\
S_4 &= 0 + \frac{\ln(2)}{4} + \frac{\ln(3)}{9} + \frac{\ln(4)}{16} = \frac{36\ln(2) + 16\ln(9) + 9\ln(4)}{144}
\end{aligned}$$

At this point, there is not a clear pattern to write a general formula S_k. Therefore we will need to use some additional strategies described in Sections 8.2-8.6 of the textbook, as well as in the "Infinite Series Tests" worksheet found in the same place you accessed this worksheet.