Math 1452: Partial Fraction Decomposition

At this point in the course, you may be encountering integrals that look very similar to each other, but use different integral methods to obtain a solution. For example, consider the following integrals:

$$\int \frac{5}{x^2+16}dx \qquad \int \frac{5x}{x^2+16}dx \qquad \int \frac{5x^2}{x^2+16}dx$$

Each of these integrals is solved using a different method. The first uses the antiderivative rule for $\tan^{-1}(x)$, the second uses u-substitution with $u = x^2 + 16$, and the third uses a method called *partial fraction decomposition.*

What is partial fraction decomposition? Partial fraction decomposition is the process of splitting one fraction into multiple fractions that add or subtract to give us what we started with.

How do I use partial fraction decomposition? To evaluate an integral using partial fraction decomposition, we first need to factor the denominator, and then decide which of four ways our denominator factors:

1. Distinct Linear Factors – distinct terms with only degree 1
2. Repeated Linear Factors – non-distinct terms with only degree 1
3. Distinct Quadratic Factors – distinct terms with degree 1 or 2
4. Repeated Quadratic Factors – non-distinct terms with degree 1 or 2

We will look at the partial fraction decomposition method for each of these four cases:

Ex. 1. Evaluate $\int \frac{1}{(x-2)(x-3)}dx$. We rewrite $\frac{1}{(x-2)\,(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$ linear linear	**Ex. 2.** Evaluate $\int \frac{x}{(x+1)(x+2)^2}dx$. We rewrite $\frac{x}{(x+1)\,(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$ linear repeated linear
Ex. 3. Evaluate $\int \frac{x^2-1}{x(x^2+1)}dx$. We rewrite $\frac{x^2-1}{x\,(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$ linear quadratic	**Ex. 4.** Evaluate $\int \frac{8x^3+13x}{(x^2+2)^2}dx$. We rewrite $\frac{8x^3+13x}{(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2}$ repeated quadratic

As you can see from the decompositions above, linear factors decompose with a constant numerator, like A, B, or C in the top row, and quadratic factors decompose with a linear numerator, like $Bx+C$ or $Ax+B$ in the bottom row. Additionally, when we have a repeated factor in our denominator, we need to include a term that with all possible powers of that factor, as seen in the righthand column.

To evaluate the integral, we need to solve for A, B, and C. This uses a strategy from algebra called *solving a system of equations*. To do this, we plug in strategic values of x (the values that make the denominator zero) and solve the equations that come out.

Ex. 1. Evaluate $\int \frac{1}{(x-2)(x-3)}dx$.

We rewrite

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$\downarrow$$

$$\begin{aligned} 1 &= A(x-3) + B(x-2) \\ x=3: 1 &= A\cdot(3-3) + B\cdot(3-2) \\ 1 &= A\cdot 0 + B\cdot 1 \\ 1 &= B \\ x=2: 1 &= A\cdot(2-3) + B\cdot(2-2) \\ 1 &= A\cdot -1 + B\cdot 0 \\ -1 &= A \end{aligned}$$

$$\downarrow$$

$$\frac{1}{(x-2)(x-3)} = \frac{-1}{x-2} + \frac{1}{x-3}$$

Ex. 2. Evaluate $\int \frac{x}{(x+1)(x+2)^2}dx$.

We rewrite

$$\frac{x}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$\downarrow$$

$$\begin{aligned} x &= A(x+2)^2 + B(x+1)(x+2) + C(x+1) \\ x=-2: -2 &= A\cdot 0 + B\cdot -1\cdot 0 + C\cdot -1 \\ -2 &= -C \\ 2 &= C \\ x=-1: -1 &= A\cdot(-1+2)^2 + B\cdot 0 + C\cdot 0 \\ -1 &= A \\ \underset{2=C,\ -1=A}{x=0}: 0 &= -1\cdot 4 + B\cdot 1\cdot 2 + 2\cdot 1 \\ 0 &= 2B - 2 \\ 2 &= 2B \\ 1 &= B \end{aligned}$$

$$\downarrow$$

$$\frac{x}{(x+1)(x+2)^2} = \frac{-1}{x+1} + \frac{1}{x+2} + \frac{2}{(x+2)^2}$$

When our numerator is more complicated, instead of plugging in values for x, we can expand our expressions and equate coefficients. To do this, we look at the coefficients of each power of x (1, x, x^2, x^3, etc.) on both sides of the equation and set them equal to each other.

Ex. 3 Evaluate $\int \frac{x^2-1}{x(x^2+1)}dx$.

We rewrite

$$\frac{x^2-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\downarrow$$

$$\begin{aligned} x^2 - 1 &= A(x^2+1) + (Bx+C)x \\ &= (A+B)x^2 + Cx + A \end{aligned}$$

$\downarrow$ equate coefficients

$$\begin{aligned} -1 &= A \\ 0 &= C \\ 1 &= A + B = -1 + B \to 2 = B \end{aligned}$$

$$\downarrow$$

$$\frac{x^2-1}{x(x^2+1)} = \frac{-1}{x} + \frac{2x}{x^2+1}$$

Ex. 4 Evaluate $\int \frac{8x^3+13x}{(x^2+2)^2}dx$.

We rewrite

$$\frac{8x^3+13x}{(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2}$$

$$\downarrow$$

$$\begin{aligned} 8x^3 + 13x &= (Ax+B)(x^2+2) + (Cx+D) \\ &= Ax^3 + Bx^2 + (2A+C)x + (2B+D) \end{aligned}$$

$\downarrow$ equate coefficients

$$\begin{aligned} 8 &= A \\ 0 &= B \\ 132A + C &= 16 + C \to -3 = C \\ 0 &= 2B + D = D \end{aligned}$$

$$\downarrow$$

$$\frac{8x^3+13x}{(x^2+2)^2} = \frac{8x}{x^2+2} + \frac{-3x}{(x^2+2)^2}$$

At this point, each term can be integrated using either u-substitution with u as the linear or quadratic term in the denominator or using an antidifferentiation rule.