Math 1452: Integration by Parts

What is integration by parts? Let's recall the definition of integration by parts:

> If $u(x)$ and $v(x)$ are functions with continuous derivatives, then
>
> $$\int udv = uv - \int vdu$$

How do I know which part of my integral is the u and which is the dv? Once you identify that an integral will use integration by parts, there is a heirarchy we can use to identify which part of our integral is the u:

L ogarithm
I nverse Trig
P olynomial
E xponential
T rig

To identify u, we go down the list and check if our integrand has that type of function. If so, we set that function to be u and the rest of the integrand becomes dv. In the following examples, we identify u and dv for the integrals listed:

$\int xe^x dx$	$\int x^2 \ln(x) dx$	$\int \arcsin(x) dx$	$\int x \sin(x) dx$
$u = x$ polynomial	$u = \ln(x)$ logarithm	$u = \arcsin(x)$ inverse trig	$u = x$ polynomial
$dv = e^x dx$ exponential	$dv = xdx$ polynomial	$dv = dx$	$dv = \sin(x)dx$ trig

In each of the examples above, we can integrate the identified dv using an antidifferentiation rule. This is called *basic* integration by parts and is one of three types of integration by parts, listed below:

1. Basic: use antidifferentiation on $\int vdu$
2. Repeated: use integration by parts on $\int vdu$
3. Circular: use integration by parts on $\int vdu$ and return to the original integral $\int udv$

We work an example of each type of integration by parts and then go on to discuss how to apply integration by parts to a definite integral.

Basic Integration by Parts:

> **Example 1.** Evaluate the integral $\int x \sin(x) dx$ using integration by parts.
>
> $u = x \qquad dv = \sin(x)dx$
>
> $du = dx \qquad \int dv = \int \sin(x)dx$
>
> $v = -\cos(x)$
>
> $$\int udv = uv - \int vdu$$
>
> $$\begin{aligned}\int x\sin(x)dx &= -x\cos(x) - \int -\cos(x)dx \\ &= -x\cos(x) + \int \cos(x)dx \\ &= -x\cos(x) + \sin(x) + C\end{aligned}$$

Repeated Integration by Parts:

Example 2. Evaluate the integral $\int x^2 \sin(x)dx$ using integration by parts.

$$u = x^2 \qquad dv = \sin(x)dx$$
$$du = 2xdx \qquad \int dv = \int \sin(x)dx$$
$$v = -\cos(x)$$

$$\int udv = uv - \int vdu$$
$$\int x^2 \sin(x)dx = -x^2\cos(x) - \int -2x\cos(x)dx$$
$$= -x^2\cos(x) + \int 2x\cos(x)dx$$

Now we use integration by parts again.

$$u = 2x \qquad dv = \cos(x)dx$$
$$du = 2dx \qquad \int dv = \int \cos(x)dx$$
$$v = \sin(x)$$

$$\int udv = uv - \int vdu$$
$$\int 2x\cos(x)dx = 2x\sin(x) - \int 2\sin(x)dx$$
$$= 2x\sin(x) - (-2\cos(x)) + C$$
$$= 2x\sin(x) + 2\cos(x) + C$$

Therefore we have

$$\int x^2\sin(x)dx = -x^2\cos(x) + \int 2x\cos(x)dx$$
$$= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$$
$$= 2x\sin(x) + (2 - x^2)\cos(x) + C$$

Circular Integration by Parts:

Example 3. Evaluate the integral $\int e^x \sin(x)dx$ using integration by parts.

$$u = e^x \qquad dv = \sin(x)dx$$
$$du = e^x dx \qquad \int dv = \int \sin(x)dx$$
$$v = -\cos(x)$$

$$\int udv = uv - \int vdu$$
$$\int e^x\sin(x)dx = -e^x\cos(x) - \int -e^x\cos(x)dx$$
$$= -e^x\cos(x) + \int e^x\cos(x)dx$$

Now we use integration by parts again.

$$u = e^x \qquad dv = \cos(x)dx$$
$$du = e^x dx \qquad \int dv = \int \cos(x)dx$$
$$v = \sin(x)$$

$$\int udv = uv - \int vdu$$
$$\int e^x\cos(x)dx = e^x\sin(x) - \int e^x\sin(x)dx$$

We have arrived back at our original integral, so the rest of this question is solved using algebra.

$$\int e^x\sin(x)dx = -e^x\cos(x) + e^x\sin(x) - \int e^x\sin(x)dx + C$$
$$+\int e^x\sin(x)dx \qquad\qquad +\int e^x\sin(x)dx + C$$
$$2\int e^x\sin(x)dx = -e^x\cos(x) + e^x\sin(x) + C$$
$$\int e^x\sin(x)dx = \frac{-e^x\cos(x) + e^x\sin(x)}{2} + C$$