Math 1452: Integration by Parts on Definite Integrals

What is integration by parts? Let's recall the definition of integration by parts:

> If $u(x)$ and $v(x)$ are functions with continuous derivatives, then
> $$\int udv = uv - \int vdu$$

How do I apply integration by parts to a definite integral? Once we apply integration by parts to an indefinite integral, we evaluate *both* the uv and the $\int vdu$.

> If $u(x)$ and $v(x)$ are functions with continuous derivatives, then
> $$\int_a^b udv = uv|_a^b - \int_a^b vdu$$

Below, we will look at three examples of applying integration by parts to definite integrals, one for each type of integration by parts: basic, repeated, and circular.

Example 1. Evaluate the integral $\int_1^4 \sqrt{x}\ln(x)dx$ using integration by parts.

$$u = \ln(x) \qquad dv = \sqrt{x}dx = x^{1/2}dx$$
$$du = \frac{1}{x}dx \qquad \int dv = \int x^{1/2}dx$$
$$v = \frac{2}{3}x^{3/2}$$

$$\int udv = uv - \int vdu$$
$$\begin{aligned}\int \sqrt{x}\ln(x)dx &= \frac{2}{3}x^{3/2}\ln(x) - \int \frac{2}{3}\cdot\frac{x^{3/2}}{x}dx \\ &= \frac{2}{3}x^{3/2}\ln(x) - \frac{2}{3}\int x^{1/2}dx \\ &= \frac{2}{3}x^{3/2}\ln(x) - \frac{4}{9}x^{3/2}\end{aligned}$$

Now we evaluate *both* terms

$$\begin{aligned}\left(\frac{2}{3}x^{3/2}\ln(x) - \frac{4}{9}x^{3/2}\right)\Bigg|_1^4 &= \left(\frac{2}{3}\cdot 4^{3/2}\ln(4) - \frac{4}{9}\cdot 4^{3/2}\right) - \left(\frac{2}{3}\cdot 1^{3/2}\ln(1) - \frac{4}{9}\cdot 1^{3/2}\right) \\ &= \frac{16}{3}\ln(4) - \frac{32}{9} + \frac{4}{9} \\ &= \frac{16}{3}\ln(4) - \frac{28}{9}\end{aligned}$$

Example 2. Evaluate the integral $\int_0^1 x^2e^xdx$ using integration by parts.

$$u = x^2 \qquad dv = e^xdx$$
$$du = 2xdx \qquad \int dv = \int e^xdx$$
$$v = e^x$$

$$\int udv = uv - \int vdu$$
$$\int x^2e^xdx = x^2e^x - \int 2xe^xdx$$

Now we use integration by parts again.

$$u = 2x \qquad dv = e^xdx$$
$$du = 2dx \qquad \int dv = \int e^xdx$$
$$v = e^x$$

$$\int udv = uv - \int vdu$$
$$\begin{aligned}\int 2xe^xdx &= 2xe^x - \int 2e^xdx \\ &= 2xe^x - 2e^x\end{aligned}$$

Therefore we have

$$\begin{aligned}\int_0^1 x^2e^x dx &= \left(x^2e^x - (2xe^x - 2e^x)\right)\Big|_0^1\\ &= \left(x^2e^x - 2xe^x + 2e^x\right)\Big|_0^1\\ &= \left(1^2e^1 - 2\cdot 1\cdot e^1 + 2e^1\right) - \left(0^2e^0 - 2\cdot 0\cdot e^0 + 2e^0\right)\\ &= e - 2e + 2 - 2\\ &= -e\end{aligned}$$

Example 3. Evaluate the integral $\int_0^\pi e^{2x}\cos(2x)dx$ using integration by parts.

$$u = e^{2x} \qquad dv = \cos(2x)dx \qquad \int udv = uv - \int vdu$$

$$du = 2e^{2x}dx \qquad \int dv = \int \cos(2x)dx \qquad \int e^{2x}\cos(2x)dx = e^{2x}\cos(2x) - \int e^{2x}\sin(2x)dx$$

$$v = \frac{1}{2}\sin(2x)$$

Now we use integration by parts again.

$$u = e^{2x} \qquad dv = \sin(2x)dx \qquad \int udv = uv - \int vdu$$

$$du = 2e^{2x}dx \qquad \int dv = \int \sin(2x)dx$$

$$v = -\frac{1}{2}\cos(2x)$$

$$\begin{aligned}\int e^{2x}\sin(2x)dx &= -\frac{1}{2}e^{2x}\cos(2x) - \int -e^{2x}\cos(2x)dx\\ &= -\frac{1}{2}e^{2x}\cos(2x) + \int e^{2x}\cos(2x)dx\end{aligned}$$

We have arrived back at our original integral, so the rest of this question is solved using algebra.

$$\begin{aligned}\int e^{2x}\cos(2x)dx &= e^{2x}\cos(2x) + \frac{1}{2}e^{2x}\cos(2x) - \int e^{2x}\cos(2x)dx\\ 2\int e^{2x}\cos(2x)dx &= -\frac{1}{2}e^{2x}\cos(2x)\end{aligned}$$

Now we divide both sides by 2 and evaluate using our bounds.

$$\begin{aligned}\int_0^\pi e^{2x}\cos(2x)dx &= -\frac{1}{4}e^{2x}\cos(2x)\Big|_0^\pi\\ &= \left(-\frac{1}{4}e^{2\pi}\cos(2\pi)\right) - \left(-\frac{1}{4}e^0\cos(0)\right)\\ &= -\frac{1}{4}e^{2\pi} + \frac{1}{4}\end{aligned}$$