Math 1451: Definite Integration by Substitution

In these examples, we will explore two *different* ways to evaluate definite integrals using substitution. Recall that indefinite integration by substitution is defined as follows:

> If we are given a composite function $f(x) = g(u(x))$ and if G is an antiderivative of g, then
> $$\int f(x)dx = \int g(u)\frac{du}{dx}dx = \int g(u)du = G(u) + C$$

Let's look at the example $\int (4x-5)^3 dx$.

1. Identify $f(x)$, $g(x)$, and $u(x)$

$$f(x) = (4x-5)^3$$
$$g(x) = x^3$$
$$u(x) = 4x-5$$

2. Differentiate $u(x)$

$$u(x) = 4x-5$$
$$\frac{du}{dx} = 4$$
$$du = 4dx$$
$$\frac{du}{4} = dx$$

3. Rewrite and integrate

$$\begin{aligned}
&\int (4x-5)^3 dx \\
&= \int u^3 \frac{du}{4} \\
&= \int \frac{1}{4}u^3 du \\
&= \frac{1}{4} \cdot \frac{1}{4} \cdot u^4 + C \\
&= \frac{1}{16}u^4 + C \\
&= \frac{1}{16}(4x-5)^4 + C
\end{aligned}$$

Now, the process changes slightly for definite integrals: Let's look at the example $\int_1^2 (4x-5)^3 dx$. The first two steps are the same as above, but now we much make a choice in how to rewrite the integral. We can either change the bounds using $u(x)$ or use substitution on an indefinite integral and then evaluate using the original bounds on x.

Integrate $g(u)$ with u bounds

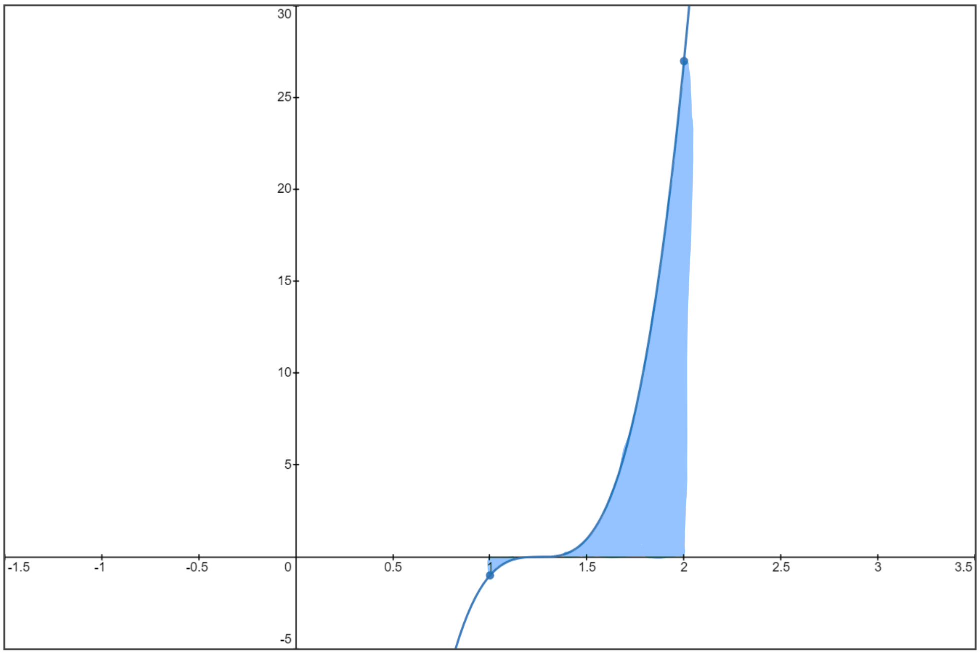
$$u(2) = 4 \cdot 2 - 5 = 3$$
$$u(1) = 4 \cdot 1 - 5 = -1$$

$$\begin{aligned}
&\int_1^2 (4x-5)^3 dx \\
&= \int_{-1}^3 u^3 \frac{du}{4} \\
&= \int_{-1}^3 \frac{1}{4}u^3 du \\
&= \frac{1}{4} \cdot \frac{1}{4} \cdot u^4 \Big|_{u=-1}^{u=3} \\
&= \frac{1}{16}u^4 \Big|_{u=-1}^{u=3} \\
&= \frac{1}{16} \cdot 3^4 - \frac{1}{16} \cdot (-1)^4 \\
&= \frac{81}{16} - \frac{1}{16} \\
&= 5
\end{aligned}$$

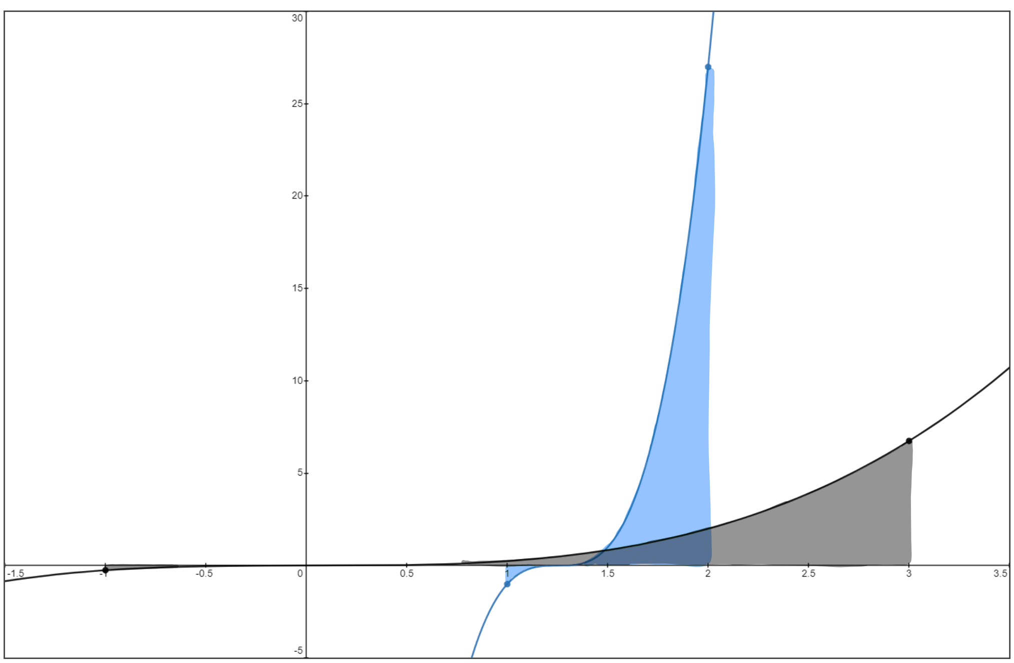
Integrate $f(x)$ with x bounds

$$\begin{aligned}
&\int (4x-5)^3 dx \\
&= \int u^3 \frac{du}{4} \\
&\vdots \quad \text{(this work is in the example above)} \\
&= \frac{1}{16}(4x-5)^4 + C \\
\\
&= \frac{1}{16}(4x-5)^4 \Big|_{x=1}^{x=2} \\
&= \frac{1}{16}(4 \cdot 2 - 5)^4 - \frac{1}{16}(4 \cdot 1 - 5)^4 \\
&= \frac{1}{16} \cdot 3^4 - \frac{1}{16} \cdot (-1)^4 \\
&= \frac{81}{16} - \frac{1}{16} \\
&= \frac{80}{16} \\
&= 5
\end{aligned}$$

What do these changes represent? Remember that integrals solve the problem of finding an area under a curve. In our example, we were calculating the area under the curve $f(x) = (4x - 5)^3$ between $x = 1$ and $x = 2$. Let's look at this graphically as the blue region shaded below.

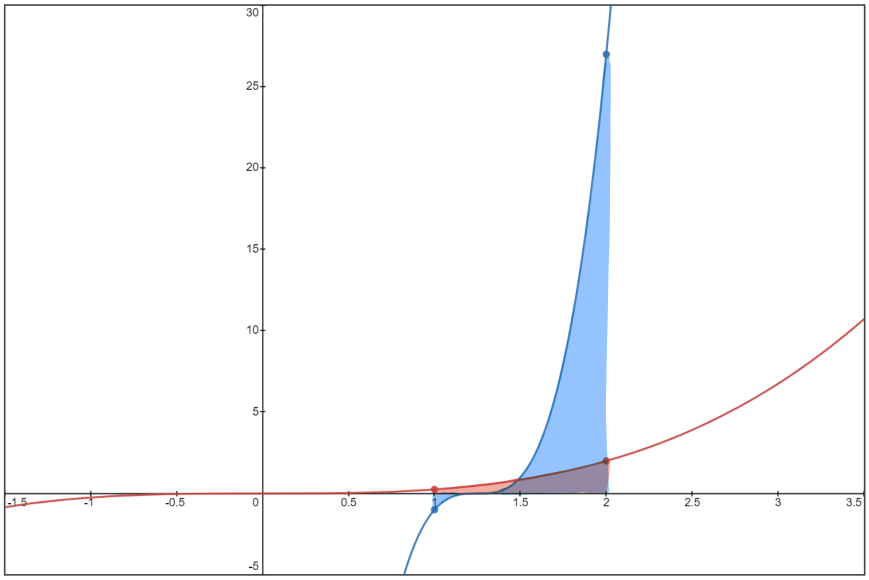


Now, once we perform the substitution and change the bounds, we are changing the integral to calculate the area under the curve $g(u) = \frac{1}{4}u^3$ between $u = -1$ and $u = 3$.



As you can see from the graphic above, the blue area and grey area are the same size. On the preceding page, the integral calculation on the left computes the green area and the integral calculation on the right computes the blue area, which are both 5.

However, if we fail to change the bounds on the integral of $g(u)$ to $u = -1$ and $u = 3$ and instead leave the bounds as $u = 1$ and $u = 2$, we are actually calculating the area of the red region below.



Based on the graph, the area of the red region is smaller than the area of the blue region, which describes our integral. To verify what we see on the graph, we calculate the area of the red region using the integral below:

$$
\begin{aligned}
& \underline{\text{Integrate } g(u) \text{ with } x \text{ bounds}} \\
& \int_1^2 (4x-5)^3 dx \\
= & \int_1^2 u^3 \frac{du}{4} \\
= & \int_1^2 \frac{1}{4} u^3 du \\
= & \frac{1}{4} \cdot \frac{1}{4} \cdot u^4 \; \Big|_{x=1}^{x=2} \\
= & \frac{1}{16} u^4 \; \Big|_{x=1}^{x=2} \\
= & \frac{1}{16} \cdot 2^4 - \frac{1}{16} \cdot 1^4 \\
= & \frac{16}{16} - \frac{1}{16} \\
= & \frac{15}{16}
\end{aligned}
$$

In this way, we can see that forgetting to change the bounds on the integral can result in a calculation far off from the correct answer.