Math 1451: Simplifying Rational Expressions

What is a rational function? A rational function is a function that is a quotient of two other functions. We will usually see this expressed as a quotient of two polynomials. Throughout calculus, we will need to simplify functions like this in order to evaluate their limit, derivative, or integral. If you haven't gotten to learn these words yet, they are the three main operations we learn in calculus.

How do we simplify rational functions? The way we simplify these functions depends on how the functions look. Sometimes, we have a simple denominator with only one or two terms, and other times, our denominator has another fraction within it. In the examples below, we look at three different structures of rational functions to simplify.

I think I am doing everything right, but something isn't working! In this case, you may be mistakenly canceling terms that do not cancel. For example, consider the *incorrect* simplification:

$$\frac{x^3-1}{x-1} = \frac{x^3 \cancel{-1}}{x \cancel{-1}} = \frac{x^3}{x} = x^2$$

Remember that a rational function is a quotient of two other functions. Simplifying in the manner above changes the function from dividing by $x-1$ to dividing by x. This is a completely different division problem! If we can remember that rational functions are a division problem and that we can only cancel out division with multiplication, then we can avoid errors like the one above.

Let's look at some simplifying strategies based on the denominator of our function.

If the denominator has a single term, we can use the strategy of *distributing division over addition and multiplication.* For example, consider the simplification:

$$\begin{aligned} f(x) &= \frac{2x^2+9x-4}{x} \\ &= \frac{2x^{\cancel{2}}}{\cancel{x}} + \frac{9\cancel{x}}{\cancel{x}} - \frac{4}{x} \\ &= 2x+9+\frac{4}{x} \\ &= 2x+9+4x^{-1} \end{aligned}$$

If the denominator has two terms, one option is to *factor the numerator and denominator to find common terms.* For example, consider the simplification:

$$\begin{aligned} f(x) &= \frac{x^3-1}{x-1} \\ &= \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}} \\ &= x^2+x+1 \end{aligned}$$

If the denominator has two terms, another option is to *perform polynomial long division.* For example, consider the simplification

$$f(x) = \frac{x^3 - 1}{x - 1} = x^2 + x + 1$$

obtained by dividing

$$\begin{array}{r}
x^2 + x + 1 \\
x - 1 \overline{\big)\, x^3 \qquad\qquad - 1} \\
\underline{-\, x^3 + x^2 \qquad\quad} \\
x^2 \qquad\quad \\
\underline{-\, x^2 + x \quad} \\
x - 1 \\
\underline{-\, x + 1} \\
0
\end{array}$$

If the denominator is a rational expression, we are dealing with a so-called *complex rational function.* To simplify this type expression, we simplify the numerator and denominator separately and then divide the results. For example, consider the simplification:

$$\begin{aligned}
f(x) &= \frac{1 + \frac{1}{x}}{3 - \frac{1}{x}} \\
&= \frac{1 \cdot \frac{x}{x} + \frac{1}{x}}{3 \cdot \frac{x}{x} - \frac{1}{x}} \\
&= \frac{\frac{x+1}{x}}{\frac{3x-1}{x}} \\
&= \frac{x + 1}{x} \div \frac{3x - 1}{x} \\
&= \frac{x + 1}{\cancel{x}} \cdot \frac{\cancel{x}}{3x - 1} \\
&= \frac{x + 1}{3x - 1}
\end{aligned}$$