Math 1451: The Second Fundamental Theorem of Calculus

What is the second fundamental theorem of calculus? This is a result that allows us to quickly take the derivative of an integral. Differentiating and integrating are mathematical inverses of each other (like addition & subtraction, multiplication & division, etc.), which means they cancel each other out and we get back the function that we started with. Formally, this is stated as:

Let $f(t)$ be continuous on the interval $[a, b]$ and define the function F by the integral equation

$$F(x) = \int_a^x f(t)dt$$

for $a \leq x \leq b$. Then F is an antiderivative of f on $[a, b]$; that is,

$$F'(x) = \frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

In practice, this means that if we are asked to take the derivative of an integral with an x as the upper bound, we just replace the integrand with x and that is our answer. For example, if we define $F(x) = \int_7^x (2t-3)dt$, then $F'(x) = 2x - 3$, obtained by replacing t in the integrand with x.

What if the integral looks different? We explore four different options below:

Ex. Differentiate $F(x) = \int_7^x (2t-3)dt$.
Solution:

$$\frac{d}{dx}(F(x)) = \frac{d}{dx}\left(\int_7^x (2t-3)dt\right)$$
$$F'(x) = 2x - 3$$

Ex. Differentiate $F(x) = \int_x^7 (2t-3)dt$.
Solution:

$$\frac{d}{dx}(F(x)) = \frac{d}{dx}\left(\int_x^7 (2t-3)dt\right)$$
$$F'(x) = \frac{d}{dx}\left(-\int_7^x (2t-3)dt\right)$$
$$F'(x) = -\frac{d}{dx}\left(\int_7^x (2t-3)dt\right)$$
$$F'(x) = -(2x-3)$$
$$F'(x) = -2x + 3$$

Ex. Differentiate $F(x) = \int_7^{x^2} (2t-3)dt$.
Solution:

$$\frac{d}{dx}(F(x)) = \frac{d}{dx}\left(\int_7^{x^2} (2t-3)dt\right)$$
$$F'(x) = (2(x^2) - 3) \cdot \frac{d}{dx}(x^2)$$
$$F'(x) = (2x^2 - 3) \cdot 2x$$
$$F'(x) = 4x^3 - 6x$$

Ex. Differentiate $F(x) = \int_{x^2}^7 (2t-3)dt$.
Solution:

$$\frac{d}{dx}(F(x)) = \frac{d}{dx}\left(\int_7^{x^2} (2t-3)dt\right)$$
$$F'(x) = \frac{d}{dx}\left(-\int_{x^2}^7 (2t-3)dt\right)$$
$$F'(x) = -\frac{d}{dx}\left(\int_{x^2}^7 (2t-3)dt\right)$$
$$F'(x) = -(2(x^2) - 3) \cdot \frac{d}{dx}(x^2)$$
$$F'(x) = -(2x^2 - 3) \cdot 2x$$
$$F'(x) = -4x^3 + 6x$$