Math 1451: Related Rates

What is a related rate? A related rate is exactly what it sounds like - a rate of change of a specific variable that is related in some way to another variable. This often comes in the form of using an area or volume formula and then applying implicit differentiation.

How do I solve related rates questions? We solve these types of questions like a traditional word problem, with an extra step of differentiation in the middle:

1. Identify our known and unknown quantities.
2. Find a formula that includes all the quantities in our question.
3. Use implicit differentiation on your formula.
4. Plug in our known quantities.
5. Solve for our unknown quantity.
6. Write a conclusion, remembering to include units.

Example 1. Consider a cone with a radius of 6 inches. Find the rate of change of the volume if the radius is increasing at a rate of 2 inches per minute and the height is three times the radius.

1. Identify our known and unknown quantities.
   "Consider a cone with a radius of 6 inches" $\rightarrow r = 6$
   "Find the rate of change of the volume" $\rightarrow \frac{dV}{dt}$ is our unknown
   "The radius is increasing at a rate of 2 inches per minute" $\rightarrow \frac{dr}{dt} = 2$
   "The height is three times the radius" $\rightarrow h = 3r = 18$

2. Find a formula that includes all the quantities in our question.
   The question tells us that we are dealing with a cone, specifically with the volume, so our formula is $V = \frac{1}{3}\pi r^2 h$.

3. Use implicit differentiation on your formula.

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(2r\frac{dr}{dt}h + \frac{dh}{dt}r^2\right) = \frac{2\pi}{3}rh\frac{dr}{dt} + \frac{\pi}{3}r^2\frac{dh}{dt}$$

4. Plug in our known quantities.

$$\frac{dV}{dt} = \frac{2\pi}{3}\cdot 6\cdot 18\cdot 2 + \frac{\pi}{3}\cdot 36\cdot\frac{dh}{dt} = 144\pi + 12\pi\frac{dh}{dt}$$

5. Solve for our unknown quantity.
   At this step, we encounter a problem. The equation above has more than just our unknown variable $\frac{dV}{dt}$, it also has $\frac{dh}{dt}$! At this point, we realize that the question statement did not explicitly tell us information about $\frac{dh}{dt}$.

What do I do when I am not given all the information in a question? When the formula we are using in a related rates question has more than two variables, there will be some relationship between the variables to make sure we only have one unknown variable to solve for. In the example above, we need to use the relationship between height h and radius r to find a relationship between $\frac{dr}{dt}$ and $\frac{dh}{dt}$.

Example 1. (continued) Since we know that $h = 3r$, we can use implicit differentiation on this formula to find that $\frac{dh}{dt} = 3\frac{dr}{dt} = 3 \cdot 2 = 6$ and finish our problem.

$$\frac{dV}{dt} = 144\pi + 12\pi \cdot 6 = 144\pi + 72\pi = 216\pi$$

As our conclusion, the rate of change of the volume is 216π cubic inches per minute.

Example 2. A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is flowing out of the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.

1. Identify our known and unknown quantities
   "A conical tank (with vertex down) is 10 feet across the top and 12 feet deep" $\rightarrow$ the relationship between the diameter and height is $12d = 10h$
   "If water is flowing out of the tank at a rate of 10 cubic feet per minute" $\rightarrow \frac{dV}{dt} = -10$
   "find the rate of change of the depth of the water" $\rightarrow \frac{dh}{dt}$ is our unknown
   "when the water is 8 feet deep" $\rightarrow h = 8$
   Like the first example, we will need to use the relationship $12d = 10h$ more than once. We rewrite this relationship as

$$\begin{aligned} 12d &= 10h \\ 12 \cdot 2r &= 10h \\ 24r &= 10h \\ r &= \frac{10}{24}h \\ r &= \frac{5}{12}h \\ \frac{dr}{dt} &= \frac{5}{12}\frac{dh}{dt} \end{aligned}$$

   Since $h = 8$, we have that $r = \frac{5}{12} \cdot 8 = \frac{40}{12} = \frac{10}{3}$.

2. Find a formula that includes all the quantities in our question
   The question tells us that we are dealing with the volume of water in a cone, so our formula is $V = \frac{1}{3}\pi r^2 h$.

3. Use implicit differentiation on your formula

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(2r\frac{dr}{dt}h + \frac{dh}{dt}r^2\right) = \frac{2\pi}{3}rh\frac{dr}{dt} + \frac{\pi}{3}r^2\frac{dh}{dt}$$

4. Plug in our known quantities

$$-10 = \frac{2\pi}{3} \cdot \frac{10}{3} \cdot 8 \cdot \frac{5}{12} \cdot \frac{dh}{dt} + \frac{\pi}{3} \cdot \frac{100}{9} \cdot \frac{dh}{dt} = \frac{100\pi}{9}\frac{dh}{dt}$$

5. Solve for our unknown quantity

$$-10 = \frac{100\pi}{9}\frac{dh}{dt} \quad \rightarrow \quad -10 \cdot \frac{9}{100\pi} = \frac{9}{100\pi} \cdot \frac{100\pi}{9}\frac{dh}{dt} \quad \rightarrow \quad -\frac{9}{10\pi} = \frac{dh}{dt}$$

6. Write a conclusion, remembering to include units.
   The rate of change of the depth of the water is $-\frac{9}{10\pi}$ feet per minute.