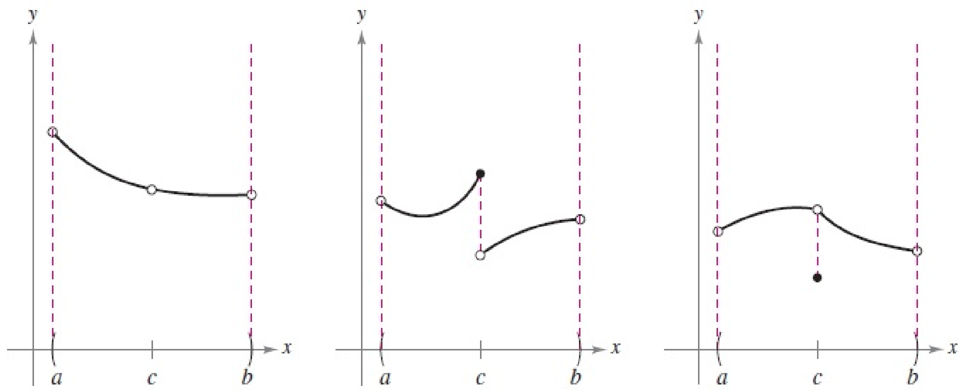
Math 1451: Continuity with Piecewise Functions

What is continuity? A function is continuous at the point $x = c$ if there is no interruption in the graph at that point, i.e., the graph is not broken and there are no holes, jumps, or gaps. The images below represent three cases in which a function is not continuous at $x = c$.



If we do not know the graph of the function, we can use analytical methods to determine continuity. An equivalent definition of continuity is that a function is continuous at $x = c$ if ...

1. $f(c)$ is defined
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

When any of these conditions fail, the function is no longer continuous. In the first graph above, $f(c)$ is not defined. In the second graph above, $\lim_{x \to c} f(x)$ does not exist. In the third graph above, $f(c)$ is defined and the limit exits, but $\lim_{x \to c} f(x) \neq f(c)$.

Which functions are continuous? Luckily for us, many of the functions we look at in calculus are continuous, such as polynomial functions, radical functions, and the trigonometric functions $\sin(x)$ and $\cos(x)$. **Which functions are *not* continuous?** Some functions that tend to not be continuous are rational functions, the trigonometric functions $\tan(x)$, $\cot(x)$, $\sec(x)$, and $\csc(x)$, and piecewise functions. In this worksheet, we will look specifically at piecewise functions.

What questions may I be asked about continuity of piecewise functions? There are two main question types you will be asked about continuity of piecewise functions:

1. Stating values of x at which the function is not continuous.
2. Solving for a variable a that makes a piecewise function continuous.

For these questions, it is important to remember that if the function has different behavior to the left and right of the point $x = c$, the limit does not exist. We will make sure the limit exits by computing the lefthand limit and the righthard limit and making sure they are the same.

Example 1. Determine if the function $f(x)$ is continuous. If it is not, state the values of x at which the function is not continuous.

$$f(x) = \begin{cases} x+1, & x \leq 1 \\ x^2-1, & x > 1 \end{cases}$$

Since $x+1$ and x^2+1 are polynomial functions they are continuous for all values of x, so we only need to be concerned about continuity at $x = 1$, where the function changes between the two polynomials. We work through the three steps to check continuity:

1. Verify that $f(1)$ is defined.
   We evaluate $f(1) = 1 + 1 = 2$. ✓

2. Verify that $\lim_{x\to1} f(x)$ exists.
   To do this, we take the lefthand limit of the function and righthand limit of the function to verify that they are the same.

   $$\lim_{x\to1^-} f(x) = \lim_{x\to1} (x+1) = 1+1 = 2 \qquad \lim_{x\to1^+} f(x) = \lim_{x\to1} (x^2-1) = 1^2 - 1 = 0$$

   Since the lefthand limit and righthand limit are not the same, the limit does not exist and therefore the function is not continuous at $x = 1$.

3. Since $\lim_{x\to1} f(x)$ does not exist, we cannot compare the limit value to the function value.

Since all the second condition is not satisfied, $f(x)$ is not continuous at $x = 1$ and is continuous for all other values of x.

Example 2. Determine the value(s) of a that makes the function $f(x)$ continuous.

$$f(x) = \begin{cases} x^3, & x \leq 2 \\ ax^2, & x > 2 \end{cases}$$

Again, since x^3 and ax^2 are polynomial functions, they are continuous for all values of x, so we only need to be concerned about continuity at $x = 2$. We solve for the variable a that ensures that $f(x)$ is defined at $x = 2$ and $\lim_{x\to2} f(x)$ exists:

1. Verify that $f(2)$ is defined.
   We evaluate $f(2) = 2^3 = 8$. ✓

2. Verify that $\lim_{x\to2} f(x)$ exists.
   To do this, we take the lefthand limit of the function and righthand limit of the function to verify that they are the same.

   $$\lim_{x\to2^-} f(x) = \lim_{x\to2} (x^3) = 2^3 = 8 \qquad \lim_{x\to2^+} f(x) = \lim_{x\to2} (ax^2) = a \cdot 2^2 = 4a$$

   For these limits to be equal, we must have $8 = 4a$, or equivalently, $2 = a$. This will make $\lim_{x\to2} f(x) = 8$. ✓

3. From steps 1 and 2 above, we can see that $\lim_{x\to2} f(x) = f(2)$. ✓

Since all three conditions are satisfied when $a = 2$, this is our answer.