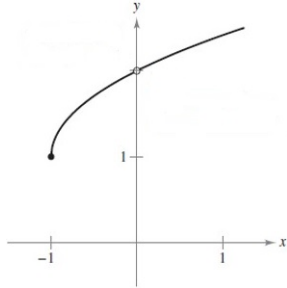# Math 1451: What is a Limit?

What is a limit? Given a function $f(x)$, a limit tells us what we *expect* the function to do at a specific x-value. For example, consider the graph below. We can see that the graph has an open circle at the point $(0, 2)$, so we say that the *function value* is undefined at $x = 0$.

However, we can trace the graph and see that at $x = 0$, we *expect* the function to be at $y = 2$. We write this with the notation

$$\lim_{x \to 0} f(x) = 2$$

and say that the function has a *limit value* of 2 at $x = 0$.

Does the function value ever match the limit value? Sometimes, but not always. When we evaluate limits, our first strategy will be to see if the function is defined at the x-value specified in the limit. If the function is undefined or indeterminate at that point, we will try another method to evaluate the limit.

Can we always find a limit? For some graphs, tracing the value of what the graph should be does not give one conclusive answer. There are three situations in which this might occur:
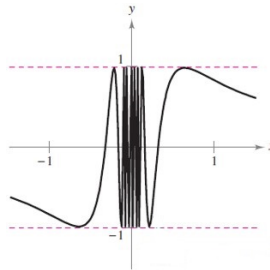
1. Different left and right behavior

2. Oscillating behavior

3. Unbounded behavior

$\lim_{x \to 0} f(x)$ does not exist
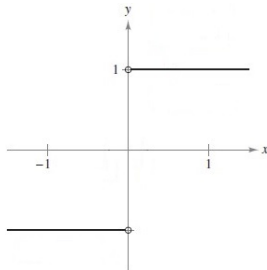
$\lim_{x \to 0} f(x)$ does not exist

$\lim_{x \to \frac{\pi}{2}} f(x)$ does not exist

Tracing the graph from the left we expect the function value to be -1 but tracing the graph from the right we expect the function value to be 1.
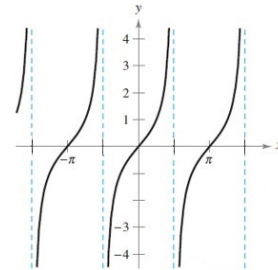
The function bounces between the values 1 and -1 as we get closer and closer to 0.

When we trace the graph going closer to $\frac{\pi}{2}$ on the left, the function keeps increasing without bound.

How do I calculate a limit when it exists? First, we should try to evaluate the function at the point specified in the limit to see if a function value exists. If no function value exists or we obtain an indeterminate value, then we can try one of the following options:

1. Simplify the function by factoring

2. Rationalize the function

3. Evaluate using the special trig limits

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 0 \quad \text{and} \quad \lim_{x \to 0} \frac{1 - \cos(x)}{x - 1} = 0$$

4. Use L'Hopital's rule. We will come back to this method later in the semester.

Example 1. Evaluate the limit
$$\lim_{x \to -3} \frac{4x^2 + 11x - 3}{x^2 - x - 12}$$
First, we evaluate the function at $x = -3$ and get
$$\frac{4(-3)^2 + 11(-3) - 3}{(-3)^2 - (-3) - 12} = \frac{36 - 33 - 3}{9 + 3 - 12} = \frac{0}{0},$$
which is an indeterminate value. To remedy this, we try and factor out $x - (-3)$ from the numerator and denominator.
$$\frac{4x^2 + 11x - 3}{x^2 - x - 12} = \frac{(4x + 1)(x + 3)}{(x - 4)(x + 3)} = \frac{(4x + 1)\cancel{(x + 3)}}{(x - 4)\cancel{(x + 3)}} = \frac{4x + 1}{x - 4}$$
Now we evaluate this new function at the point $x = -3$ and get a limit value of
$$\frac{4x + 1}{x - 4} = \frac{-12 - 1}{-3 - 4} = \frac{-13}{-7} = \frac{13}{7}.$$

Example 2. Evaluate the limit
$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1}$$
First, we evaluate the function at $x = 1$ and get
$$\frac{\sqrt{1} - 1}{1 - 1} = \frac{0}{0},$$
which is an indeterminate value. To remedy this, we rationalize the fraction by multiplying by the conjugate of the numerator.
$$\frac{\sqrt{x} - 1}{x - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} = \frac{\sqrt{x^2} + \cancel{\sqrt{x}} - \cancel{\sqrt{x}} - 1}{(x - 1)(\sqrt{x} + 1)} = \frac{\cancel{x - 1}}{\cancel{(x - 1)}(\sqrt{x} + 1)} = \frac{1}{\sqrt{x} + 1}$$
Now we evaluate this new function at the point $x = 1$ and get a limit value of
$$\frac{1}{\sqrt{1} + 1} = \frac{1}{2}.$$

Example 3. Evaluate the limit
$$\lim_{x \to 0} \frac{\sin^2 x}{2x}$$
First, we evaluate the function at $x = 0$ and get
$$\frac{0^2}{2 \cdot 0} = \frac{0}{0},$$
which is an indeterminate value. To remedy this, we split the fraction into a product of two different fractions.
$$\frac{\sin^2 x}{2x} = \frac{\sin(x) \cdot \sin(x)}{2 \cdot x} = \frac{\sin(x)}{2} \cdot \frac{\sin(x)}{x}$$
Now we recall that the limit of a product is the product of the limits and get a limit value of
$$\lim_{x \to 0} \frac{\sin^2 x}{2x} = \lim_{x \to 0} \frac{\sin(x)}{2} \cdot \lim_{x \to 0} \frac{\sin(x)}{x} = \frac{0}{2} \cdot 1 = 0 \cdot 1 = 0.$$