Math 1451: Finding Extrema

There are two different types of extrema: relative and absolute. Relative extrema occur within a given interval of a graph and absolute extrema occur within the entire graph. There are three different ways a question may ask you to find the extrema of a function.

When the question asks you to find relative extrema inside an interval...

1. Find the critical points by solving the equation $f'(x) = 0$.
2. Check if the critical points are inside the interval. If they fall outside of the interval, we do not consider them.
3. Evaluate the function at each critical point inside the interval and at each endpoint of the interval.
4. The smallest value represents an absolute minimum and the largest value represents an absolute maximum.

When the question asks you to find absolute extrema using the first derivative test...

1. Find the critical points by computing the first derivative.
2. Plot the critical points on a number line and find test values between each critical point.
3. Evaluate the derivative at each test value to determine where the function is increasing ($f'(x) > 0$) or decreasing ($f'(x) < 0$).
4. Determine the extrema using the following criteria:
   (a) If the function changes from increasing to decreasing at a critical point, the critical point is a maximum.
   (b) If the function changes from decreasing to increasing at a critical point, the critical point is a minimum.
   (c) If the function stays decreasing or stays increasing at a critical point, the critical point is a neither a minimum or a maximum.

When the question asks you to find absolute extrema using the second derivative test...

1. Find the critical points by computing the first derivative.
2. Compute the second derivative.
3. Evaluate the second derivative at each critical point to determine where the function is concave up ($f''(x) > 0$) or concave down ($f''(x) < 0$).
4. Determine the extrema using the following criteria:
   (a) If the function is concave down, the critical point is a maximum.
   (b) If the function is concave up, the critical point is a minimum.
   (c) If the function is neither concave up nor concave down, the critical point is a neither a minimum or a maximum.

Example 1. Find the relative extrema of $f(x) = x^4 - 2x^2 + 3$ on the interval $[-1, 2]$.

1. Find the critical points by solving $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 4x^3 - 4 \\ 4x^3 - 4x &= 0 \\ 4x(x^2 - 1) &= 0 \\ 4x(x-1)(x+1) &= 0 \end{aligned}$$

$$\begin{array}{ccc} 4x = 0 & x - 1 = 0 & x + 1 = 0 \\ x = 0 & x = 1 & x = -1 \end{array}$$

2. All the critical points of the function fall within the interval $[-1, 2]$.

3. Evaluate the function at each critical point and at each endpoint of the interval.

$$\begin{aligned} f(0) &= 0^4 - 2 \cdot 0^2 + 3 = 3 \\ f(1) &= 1^4 - 2 \cdot 1^2 + 3 = 2 \\ f(-1) &= (-1)^4 - 2 \cdot (-1)^2 + 3 = 2 \\ f(2) &= 2^4 - 2 \cdot 2^2 + 3 = 11 \end{aligned}$$

4. The least value is 2, so $(-1, 2)$ and $(1, 2)$ are absolute minimums. The greatest value is 11, so $(2, 11)$ is an absolute maximum.

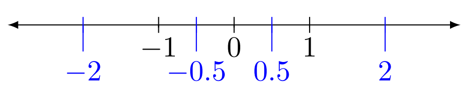
Example 2. Find the absolute extrema of $f(x) = x^4 - 2x^2 + 3$ using the first derivative test.

1. Find the critical points by solving $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 4x^3 - 4x \\ 4x^3 - 4x &= 0 \\ 4x(x^2 - 1) &= 0 \\ 4x(x-1)(x+1) &= 0 \end{aligned}$$

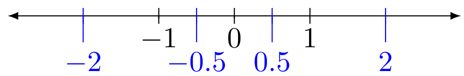
$$\begin{array}{ccc} 4x = 0 & x - 1 = 0 & x + 1 = 0 \\ x = 0 & x = 1 & x = -1 \end{array}$$

2. Plot the critical points on a number line and find test values between each critical point.



3. Evaluate the derivative at each test value to determine where the function is increasing or decreasing.

$$\begin{aligned} f'(-2) &= 4 \cdot (-2)^3 - 4 \cdot (-2) = -24 < 0 \\ f'(-0.5) &= 4 \cdot (-0.5)^3 - 4 \cdot (-0.5) = 1.5 > 0 \\ f'(0.5) &= 4 \cdot (0.5)^3 - 4 \cdot (0.5) = -1.5 < 0 \\ f'(2) &= 4 \cdot (2)^3 - 4 \cdot (2) = 24 > 0 \end{aligned}$$



4. (a) The function changes from increasing to decreasing at $x = 0$, so the critical point $(0, 3)$ is a maximum.
   (b) The function changes from decreasing to increasing at $x = -1$ and $x = 1$, so the critical points $(-1, 2)$ and $(1, 2)$ are minimums.
   (c) There is no critical point where the function stays decreasing or stays increasing.

Example 3. Find the absolute extrema of $f(x) = x^4 - 2x^2 + 3$ using the second derivative test.

1. Find the critical points by solving $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 4x^3 - 4x \\ 4x^3 - 4x &= 0 \\ 4x(x^2 - 1) &= 0 \\ 4x(x-1)(x+1) &= 0 \end{aligned}$$

$$\begin{aligned} 4x &= 0 \qquad & x - 1 &= 0 \qquad & x + 1 &= 0 \\ x &= 0 \qquad & x &= 1 \qquad & x &= -1 \end{aligned}$$

2. Compute the second derivative $f''(x) = 12x^2 - 4$.

3. Evaluate the second derivative at each critical point to determine where the function is concave up or concave down.

$$\begin{aligned} f''(-1) &= 12 \cdot (-1)^2 - 4 = 8 > 0 \\ f'(0) &= 12 \cdot (0)^2 - 4 = -4 < 0 \\ f'(1) &= 12 \cdot (1)^2 - 4 = 8 > 0 \end{aligned}$$

4. (a) The function is concave down at $x = 0$ so the critical point $(0, 3)$ is a maximum.
   (b) The function is concave up at $x = -1$ and $x = 1$, so the critical points $(-1, 2)$ and $(1, 2)$ are minimums.
   (c) There is no critical point where the function is neither concave up nor concave down.