MAPLE Worksheet Number 7

Derivatives in Calculus

The MAPLE command for computing the derivative of a function depends on which of the two ways we used to define the function, as a symbol or as an operation. We illustrate by defining the function $\mathrm{f}(x) = (2\,x+3)^5$ in each way and computing its derivative in each case. Perform the following command sequence.

```
> f[1]:=(2*x+3)^5;0;
>
> f[2]:=x->(2*x+3)^5;
```

We can compute the derivative of the symbolic expression f_1 using the command

```
> diff(f[1],x);
```

Why do you think we need to include the ,x in this command? We can compute the derivative of the operation f_2 with the command

```
> D(f[2])(x);
```

Now, in each of these two derivative computations replace the x in the MAPLE syntax with a different variable and see what happens. Explain what's going on.

We could also have used the diff command on f_2.

```
> diff(f[2](x),x);
```

Here I bet you get 0 if you replace either of the x's with another variable and leave the other alone. Why is this?

Sometimes it will be more convenient to think of the function as a symbolic expression and other times we will prefer the operation approach. Often it will be up to you to decide which is best for your situation. (You should recall the above function is the same one as g1 in worksheet #5.) Use both approaches to compute the derivatives with respect to x of the other three functions at the end of worksheet #5. They were $g2 = \ln(x)$, $g3 = \frac{1}{x}$, and $g4 = \sin(a\,x)$.

Note, as far as MAPLE is concerned all derivatives are partial derivatives. To illustrate perform the following MAPLE command sequence.

> $g := a\,x^3\,y^2 + \sin(x\,y) + \frac{1}{x}$

> $dgx := \frac{\partial}{\partial x}\,g$

> $dgy := \frac{\partial}{\partial y}\,g$

> $\frac{\partial}{\partial y}\,dgx$

> $\frac{\partial}{\partial x}\,dgy$

Explain why the last two answers are the same.

Next let's use MAPLE to recall all the basic rules of differentiation: power rule, product rule, quotient rule, and chain rule. Define each of the functions as operations. This makes substitution for the chain rule example easier.

> $f := x \rightarrow \cos\left(\frac{x}{2}\right)$

> $u := x \rightarrow x^2 - 3\,x + 1$

> $h := x \rightarrow \sqrt{x^2 + 2\,x}$

Compute the following derivatives and state which differentiation rule, or rules, MAPLE appears to be using.

> $\frac{\partial}{\partial x}\,\mathrm{u}(x)$

> $\frac{\partial}{\partial x}\,\mathrm{f}(x)\,\mathrm{u}(x)$

> $\frac{\partial}{\partial x}\,\frac{\mathrm{f}(x)}{\mathrm{u}(x)}$

This looks like the product rule used on $\mathrm{f}(x)\,\frac{1}{\mathrm{u}(x)}$. Try the next command.

```
> normal(%);
```

This looks like the quotient rule with the numerator expanded out.

> $\frac{\partial}{\partial x}\,\mathrm{f}(\mathrm{u}(x))$

> $\frac{\partial}{\partial u}\,\mathrm{f}(u)$

> $\frac{\partial}{\partial x}\,\mathrm{u}(x)$

```
> %*%%;
> subs(u=u(x),%);
```

> $\frac{\partial}{\partial x}\,\mathrm{u}(\mathrm{h}(x))$

The second derivative can be computed either of two ways, one is simply to compute the derivative of the derivative , try it on f(x). Also check out the symbolic version of the input.

```
> diff(diff(f(x),x),x);
```

The other is to use a shortcut syntax as follows and check out its symbolic version.:

```
> diff(f(x),x$2);
```

The latter has the advantage that it is just as easy to compute 15 derivatives via:

```
> diff(f(x),x$15);
```

Compute the second and third derivatives of f, u, and h above.
Next try the following commands.

```
> dd2h:=diff(h(x)*f(x),x$2);
> diff(f(u(x)),x$3);
```

Here's a problem that used to be on every standard calculus test known to mankind. First try doing it by hand to see how well you would have done on such a test. (Thank goodness for our technology.)
Compute

> $\frac{\partial}{\partial x}\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{x^2+1}}}}$

GRAPHS AND DERIVATIVES. What is the basic relation between the graph of a function and the derivative of the same function?
What is the basic relation between the graph of a function and the second derivative of the same function?

Find all intercepts, local maxima, local minima, and inflection points of the following functions. Also comment on any asymptotic behavior and sketch the graphs of the function, its first derivative, and its second derivative on the same coordinate axes. Label on the printout which graph is which.

> $p1 := 2\,x^4 - 4\,x^3 - 11\,x^2 + 8\,x + 4$

> $p2 := x^5 - 5\,x^4 + 5\,x^3 + 7$

> $r1 := \frac{x^2 - 2\,x}{x^2 + 4}$

> $r2 := \frac{x^3 + x - 5}{x^2 - 3}$

Another type of differentiation we encountered in Calculus I is "implicit differentiation." Recall that the word "implicit" is used to refer to a relation between the variables which does not usually yield one variable explicitly as a function of the others, for example

> $rel := x^3\,y + y^2 - x^2 + x\,y = 3$

is such a relation. Obviously any given value for y puts some kind of restriction on the possible values x can have. But the restriction is not explicitly given. Plot the graph of rel. (Recall implicitplot.)
Notice the two pieces, called "branches" of this graph. At each point on the graph there is a tangent line and its slope is $\frac{\partial}{\partial x}y$. We use implicit differentiation to compute these slopes.

```
> drel:=implicitdiff(rel,y,x);
```

Puting the y before the x in this command tells MAPLE to treat y as a function of x. We could have just as easily told MAPLE to treat x as a function of y and computed $\frac{\partial}{\partial y}x$.
Find the slopes of the tangent lines to this graph (thinking of y as a function of x) at the point x=0 , $y=\sqrt{3}$ and the point x=0, $y=-\sqrt{3}$. You can use the subs command with two or more variables, just make sure the expression into which the substitution is to be made is the last entry in the command.

Plot the graphs of rel and these two tangent lines on the same axes. (Recall that when using implicitplot you must enter the expression to be plotted as a relation, even if it is an explicit function. So to plot the function f(x) using implicitplot you have to enter it as y=f(x). This how you will get the graphs of the two tangent lines on the picture.

In the above we have concentrated mostly on the use of the derivative to give information about the shape of the graph of the original function. However, this last bit about tangent lines leads to another, and probably more important, use of derivatives: to approximate functions with lines. To begin we approximate a differentialble function by its tangent line. Recall the function f $:x \to \cos\left(\frac{x}{2}\right)$ defined above and perform the following command sequence.

> $dqf := \dfrac{\mathrm{f}(x) - (\mathrm{f}(\pi) + \mathrm{D}(f)(\pi)\,(x-\pi))}{x-\pi}$

> $\lim_{x \to \pi} dqf$

```
> numer(dqf);
```

Notice that the numerator of dqf is f(x) minus the value of a line at x. This line is the tangent line to the graph of f(x) at the point $(\pi, \mathrm{f}(\pi))$. That $\lim_{x \to \pi} dqf = 0$ is equivalent to the limit of the difference quotient for f equalling the derivative of f. It also shows that the values of f are so close to those of the tangent line when x is close to π the quotient is small even though the denominator is going to 0! This is exactly what we mean when we say the tangent line approximates the graph of f(x) near $(\pi, \mathrm{f}(\pi))$. The equation of the tangent line is called the "linearization of f" in Calculus I.

Sketch the graph of f along with this tangent line on the same axes.

In each of the following sketch the graph of the function and its tangent line at the point with x=1 on the same axes:

$$\sqrt{x+2}\ ,\ \ln(x)\ ,\ \mathbf{e}^{(2\,x)}\ .$$

In the multiple variable case the tangent line is replaced with the tangent plane. It is defined much the same way and approximates the graph for much the same reason. The tangent plane to the graph of $z = \mathrm{f}(x, y)$ at the point (x_0, y_0)
is given by the equation

$$z = \mathrm{f}(x_0, y_0) + \left(\frac{\partial}{\partial x} f\right)(x_0, y_0)\,(x - x_0) + \left(\frac{\partial}{\partial y} f\right)(x_0, y_0)\,(y - y_0)\,.$$

Plot the graph of $z = -x^2 + y^2 + 3$ and its tangent plane at (0,1) on the same axes.

A better approximation to a curved graph would probably be a polynomial approximation, such is called the Taylor Polynomial approximation to the graph of f at the point (x_0, y_0). Try the following command sequence.

```
> taylor(sin(x),x=0,2);
> t1:=convert(%,polynom);
> taylor(sin(x),x=0,3);
```

```
> taylor(sin(x),x=0,4);
> t3:=convert(%,polynom);
> taylor(sin(x),x=0,10);
> t9:=convert(%,polynom);
```

On the same axes sketch the graphs of y=sin(x), and its taylor polynomials of orders 1, 3, and 9. Use the range $x = -2\,\pi\,..\,2\,\pi$ and $y = -2\,..\,2$. and label each function on the printout.

Compute the taylor polynomials of orders 1, 2, 3, 4 for $\mathbf{e}^x$ centered at x=1. Sketch the graphs of all these on the same axes. Label the graphs on the printout. Note that the Taylor Polynomial of order 1 is just the equation for the tangent line.

Actually MAPLE has certain common general constructs built into it. To see some examples consider the following commands. Note that here k(x) has NOT BEEN EXPLICITELY DEFINED, it is just general funcation notation.

> $\lim_{x \to a} \frac{\mathrm{k}(x) - \mathrm{k}(a)}{x - a}$

> $\lim_{x \to a} \frac{\mathrm{k}(x) - (\mathrm{k}(a) + \mathrm{D}(k)(a)\,(x-a))}{x - a}$

> $\mathrm{taylor}(\mathrm{k}(x), x = a, 7)$

What does the expression $\mathrm{O}((x-a)^7)$ represent? (To answer this question look at the Taylor expansions for sin(x), taylor(sin(x),x=0,n), for several different values of n. Then if you're nice in class I might show you a fancy "general" limit calculation that gives some indication of what it represents.)

MAPLE can compute multivariable taylor approximations also. But first we must invoke a special command as follows:

```
> readlib(mtaylor):
```

Now let's compute the 1st order Taylor approximation to $z = -x^2 + y^2 + 3$. Of course this should be the same as the equation of the tangent plane, so try it at the point (0,1) and compare with your previous result. Is it the same?

```
> mtaylor(-x^2+y^2+3,[x=0,y=1],2);
```

Next try,

`> mtaylor(-x^2+y^2+3,[x=0,y=1],3);` but first try to guess the answer. Think! You want a degree 2 polynomial in x and y that approximates the graph.

```
> simplify(%);
```

Compute the degree 1, 2, and 3 Taylor approximations to $z = \mathbf{e}^{(x^2+y^2)}$ at (0,0). Plot the graphs of z with each of these approximations on three graphs.

Experiment to see the extent to whcih it appears that MAPLE can derive a "general" formula for the

Taylor Polynomial for a given function of two or more variables.

To have a Taylor Approximation at a point the function must be differentiable at the point.
Let's look at some functions that are continuous at x=a but not differentiable at x=a. The first is gotten with the MAPLE command abs . Plot the graph of y=abs(x) with x= -10..10 and see if it is a familiar function to you.

```
> plot(abs(x),x=-10..10);
```

To see if it is continuous at x=0 compute the limit as x->0 and compare the result with abs(0).

```
> limit(abs(x),x=0);abs(0);
```

To see if abs is differentiable at 0 compute the limit of its difference quotient at x=0.

> $\lim_{x \to 0} \frac{|x| - |0|}{x - 0}$

Try approaching 0 from the left and from the right. What happens in each case? Is this consistent with the above graph near 0? Of course abs(x) is the absolute value of x and is usually denoted by $|x|$.

For each of these functions, show the function is continuous at x=0, then test to see which are, and which are not, differentiable at x=0.

$$|x^3| , \qquad |\sin(x)| , \qquad \mathrm{piecewise}\left(x = 0, 0, x \sin\left(\frac{1}{x}\right)\right) , \qquad \mathrm{piecewise}\left(x = 0, 0, x^2 \sin\left(\frac{1}{x}\right)\right)$$

We finish by looking at a function that is differentiable but not twice differentiable at x=0. Consider the function

$$\mathrm{piecewise}(x \le 0, x^2, x^3) \ .$$

Plot it for x=-2..2 Show that it's differentiable at x=0 with derivative =0. Plot its derivative.
Now show that its derivative is CONTINUOUS but NOT DIFFERENTIABLE at x=0.