IUMPST: The Journal, Vol 3 (Technology), August 2005. [www.k-12prep.math.ttu.edu]

CREATING TWO-DIMENSIONAL NETS OF THREE-DIMENSIONAL SHAPES USING *GEOMETER'S SKETCHPAD*

Dr Paula Maida
Department of Mathematics
Western Connecticut State University
mailap@wcsu.edu

Abstract

This article is about a computer lab project in which prospective teachers used Geometer's Sketchpad software to create two-dimensional nets for three-dimensional shapes. Since this software package does not contain ready-made tools for creating non-regular or regular polygons, the students used prior knowledge and geometric facts to create their own custom tools with Sketchpad. Using their handmade customized tools, students created various nets for prisms, pyramids, and the Platonic solids. The author taught nets without this approach and then with this approach, and found not only a positive difference in performance on assessments, but also more excited and engaged attitudes about the material. The pre-service teachers were eager to simulate this activity with their future students as well as use their tools for their own test-making needs.

Introduction

As proposed in NCTM's *Principle and Standards for School Mathematics* (2000),

> "Students' skills in visualizing and reasoning about spatial relationships are fundamental in geometry. Some students may have difficulty finding the surface area of three-dimensional shapes using two-dimensional representations because they cannot visualize the unseen faces of the shapes. Experience with models of three-dimensional shapes and their two-dimensional "nets" is useful in such visualization….Students should build three-dimensional objects from two-dimensional representations". (p. 237)

Through past experiences teaching college mathematics preparation courses for prospective schoolteachers, the author recognized the need for strengthening skills of visualizing three-dimensional solids from various two-dimensional nets. These pre-service teachers (hereinafter, named students) had difficulty making connections between a flat cutout and its respective folded-up solid. Burris (2005) suggests, "Prior to being formally introduced to the idea of a net, children can use their visual skills and logical reasoning to determine whether a figure was produced by folding or curling a page of paper." (p. 264) While this may be the case for elementary figures, it is not as natural when the figures become more complex. Even if students did recognize that a net created a heptagonal prism, for example, many could not further suggest where one of the rectangles should be moved to prevent the flat figure from still folding into a heptagonal prism.

The use of the plastic hands-on *Polydron* manipulative caused improvement in understanding and assessments but not with sufficient results. The author was brainstorming ways to enrich students' experiences with nets and recognized the ability to create 'custom tools' using *Geometer's Sketchpad*, a dynamic geometry software package that the students had been using once a week for the prior five weeks. A lab was consequently designed to utilize this customized option, ultimately requiring useful and significant further exposure to this material. Connecting the nets material with software usage is supported by NCTM's claim, "Students also need to examine, build, compose, and decompose complex two- and three-dimensional objects, which they can do with a variety of media, including … dynamic geometry software." (NCTM, 2000, p. 237)

Prior *Geometer's Sketchpad* Knowledge

The students completed four *Geometer's Sketchpad* labs prior to this net-making lab. During their first *Sketchpad* lab, students became immersed in the software, learning how to use its tools while creating a real-life scene that had to include sixteen particular shapes (including a circle, non-right isosceles triangle, concave quadrilateral, non-rectangular parallelogram, equilateral triangle, isosceles trapezoid, regular hexagon, and others.) The text-writing tool was used to label the shapes. The students were encouraged to do their best with creating the regular polygons, recognizing that their first attempt with dragging the line segments would not result in congruent sides and vertex angles of the polygons -- we agreed to be content with this for the meantime. In the second lab, they learned how to (among other things) correct this issue by learning how to create a regular pentagon. (The process taught is similar to that used on the attached lab sheet instructions). Throughout subsequent labs, students became proficient with their skills in rotation, reflection, and translation by a marked vector (offering movement in a particular direction, by a fixed distance).

Designing Custom Tools with *Geometer's Sketchpad*

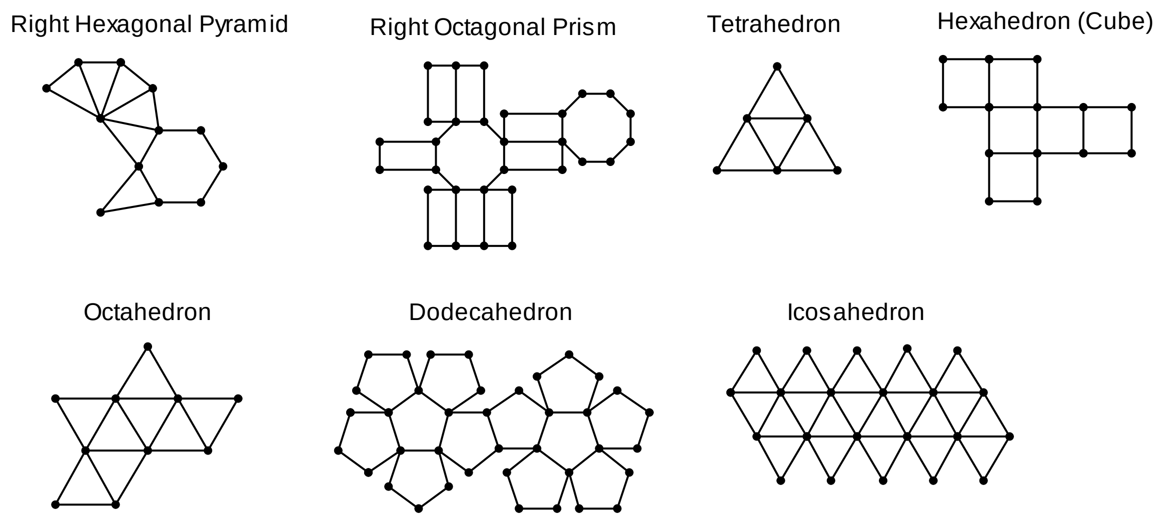
As detailed in the Nets Lab instructions at the latter part of this document, students designed custom tools to quickly create equilateral triangles, non-equilateral isosceles triangles, squares, non-square rectangles, and regular pentagons, hexagons and octagons, all using the *Sketchpad* program. Each custom tool became internal to the sketch page so that the tools could be used upon the students' choosing, during the creation of the nets. To create the custom tools for regular polygons, the students utilized geometric facts they had derived and explored in the classroom. That is, they needed to use the fact that the measure of a vertex angle in a regular n-gon is $[(n-2) \times 180^{\circ}]/n$, in order to determine the correct angle measure as they rotated line segments. This fact had been explored in the classroom, building a pattern based on minimal number of triangles inside various regular polygons and utilizing their earlier findings that angles in a triangle sum to 180°.

Creating Two-dimensional Nets for Three-dimensional Shapes

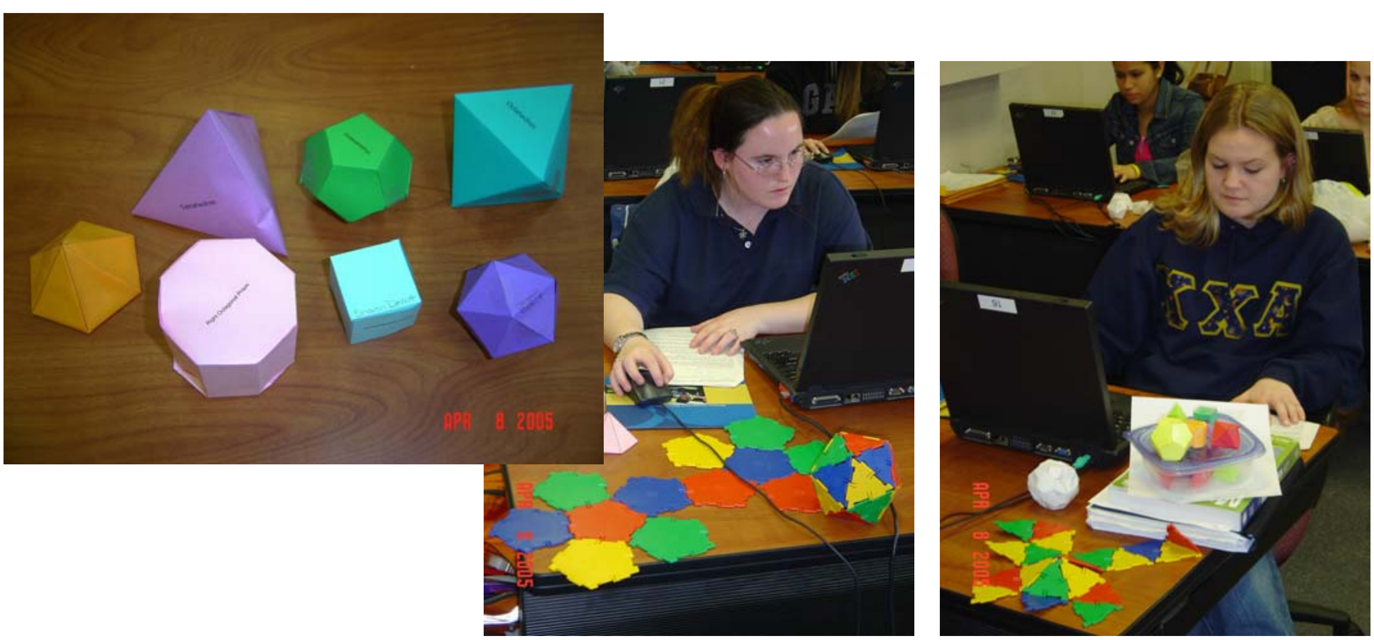
The students were instructed to use their newly designed, homemade *Sketchpad* tools to construct various two-dimensional nets for three-dimensional shapes on *Sketchpad* (again, see Nets Lab instructions below). They created right hexagonal pyramids, right octagonal prisms and the five Platonic solids: tetrahedron, cube (hexahedron), octahedron, dodecahedron, and icosahedron. For most solids, they were asked to create more than one net, which emphasized net varieties for distinct shapes. This ultimately took them out of their comfort zone of using the more 'standard' commonly seen nets of, say, pyramids and prisms.

For the construction of the right hexagonal pyramid, students used the non-equilateral isosceles triangles and they used the non-square rectangles for the right octagonal prism – they were aware that their square-making tool could have been used in that case but they opted for something different since the squares were required in the cube. Students had access to *Polydrons* in the laboratory and most students used them. *Polydrons* are a plastic manipulative of polygons that snap together to create nets and then fold up to snap into polyhedra. By working with them, students were able to troubleshoot some of their distinct "less commonly seen" nets.

Samples of students' work includes, but is certainly not limited to, the following nets:



The students printed their nets on bright colored paper, cut them out and folded them up. These pre-service teachers were impressed with their results and rightfully boasted about saving them for their future classroom use.



Exploring Euler's Formula

After the students completed their nets and folded them into three-dimensional solids, they completed the worksheet 'Exploring Euler's Formula for Polyhedra' (shown on the last page of this document). They counted and recorded the number of faces (F), vertices (V), and edges (E) for each of their polyhedra. In the final two columns of the table, the students recognized that values in the 'F + V' column were two more than results in the 'E' column. Therefore, their conjectures for Euler's formula were either $F + V = E + 2$ or $F + V - 2 = E$. Their exploration in this part of the activity led to greater impact than the author has recognized in prior years of course discussions about Euler's formula. These students were able to connect their combination of computer-generated and hand-folded work to some powerful results of 18th century mathematics.

Prospective Teachers' Reactions

Two classes, 59 students altogether, participated in this project, and were asked to comment about this lab on an informal survey. Samples of students' comments follow:

- "The nets lab helped me to identify polyhedra just by seeing a net. I really couldn't visually fold the net up until I had the chance to do it physically and learn by trial and error. That was the only way for me to see how something that looked correct on paper did not work in 3-D and it was actually kind of fun to be proven wrong and have to go make a different net that did work. Now I can look at a net and see where and how it would fold up."
- "The nets lab was helpful. I think it is more helpful in the long run because it not only helped me to remember the shapes, but it also helped us learn different formulas. For example, to explore Euler's formula, I could actually see and count the faces, vertices and edges because I had my created objects in front of me. Not only this, but when we see a net for a shape, having once folded them together, its easier to visualize the net as a shape. It also brings forth ideas of projects to do as a teacher to help kids learn about shapes."
- "The nets lab was the hardest lab this semester, and yet it is the one that stuck in my head the most. It made me think. It made me frustrated and because I was frustrated, it pushed me. It took a lot of time before I actually got them and completed them, but now I can just look at a net and see what it is and if it will work. I am now able to teach shapes and dimensions better because I have fully explored them and am able to quickly recognize them. My understanding is now deeper and I believe that this will allow me to teach that unit better. I had mixed feelings about the lab because it was so frustrating, but when I think about it, I understand that the frustration lead to the learning. If it were so easy it would not have stuck in my mind as well."
- "Working on and completing the nets lab was definitely a learning experience. Although I thought it was a bit challenging, I also had fun doing it. In the end when I was all finished, I was proud of myself for being able to figure it out. There were some nets that I thought I would never be able to figure out, like the icosahedron. There were so many sides that I kept getting confused. I definitely got frustrated at some points while designing and also putting them together

afterwards. I liked that lab and even if I might not teach that specific area, it is helpful and good to know."

- "At first when we started the net lab I thought I would do poorly because in high school I was never able to do similar problems. It was hard for me to look at a shape flat on a paper and imagine it being 3-D. I liked the lab because I am now able to understand something I never understood before. To be honest, I went home and taped all of the brightly colored shapes to my wall (they didn't stay)."

Many students stated that the lab was time-consuming but worth it and much more meaningful than simply being told results from a teacher. Claiming to be weak at visualization, they stated that this helped them hone in on their spatial skills, making it easier as they went along. A student wrote, "If it weren't for this lab, a dodecahedron would still seem more abstract and complicated than it really is." Another claimed, "Assembling polyhedra was helpful because the pictures in the book don't always help."

The students particularly commented about their improved assessment skills. Based on this project, they were able to recognize on tests the nets that worked and didn't work. Moreover, if a proposed net failed, they could defend why (e.g., "a gap here and an overlap there"). The author truly believes that a manipulative is not successful until it can be removed and a student can consequently complete a task successfully without it. The author was pleased to see that the students were able to take their experiences of what attempted nets worked and didn't work in order to attain a level of abstraction where they could make this conclusion void of a hands-on manipulative.

Conclusion

"Activities with nets enable students to transform two-dimensional figures into three-dimensional objects and back again. When students move objects between two and three dimensions they begin to build a foundation for understanding solid geometry figures that will help them in subsequent mathematics courses." (Chapin & Johnson, 2000, p. 171) The author fully concurs with Chapin and Johnson and adds that pre-service teachers benefit doubly by not only being stronger for subsequent mathematics courses but also, and more importantly, by valuing this skill with excitement for sharing it with others. The students became empowered and recognized that this activity is not only beneficial for student use, understanding and engagement, but also this net-making approach is far more efficient than the tedious net-drawing using a template of polygons for creating exams or other types of assessments. Overall, the author found this to be an effective and valuable technology lab in a mathematics preparation course for school teachers.

References

Burris, A. C. (2005). Understanding the Math You Teach: Content and Methods for Prekindergarten Through Grade 4. Upper Saddle River, NJ: Pearson Education, Inc.

Chapin, S. H. & Johnson, A. (2000). Math Matters: Understanding the Math You Teach, Grades K-6. Sausalito, CA: Math Solutions Publications.

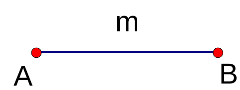
National Council of Teachers of Mathematics. (2000). Principles and Standards for School Mathematics. Reston, VA: The National Council of Teachers of Mathematics, Inc.

CREATING NETS USING *GEOMETER'S SKETCHPAD*

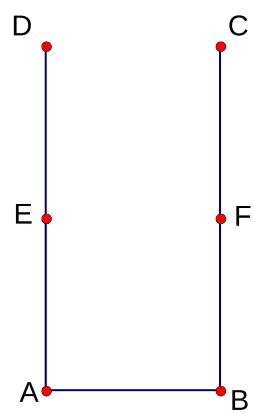
In this lab, you will construct two-dimensional nets for three-dimensional shapes. The process of adjoining many polygons to create various nets could be tedious, but we will design an efficient approach. Wouldn't it be great if the toolbar of *Geometer's Sketchpad* included tools for making rectangles and various polygons, just like it includes tools for making circles and line segments? Unfortunately, it does not include this extensive list of shape-making tools, but what's stopping us from making our own?!?

Creating a Tool to Make Non-square Rectangles

Create a horizontal line segment about one inch long. You do not need to label the segment or endpoints. However, for purposes of instruction, the segments and points shown on these lab instructions will be labeled. You have created a segment such as



Select only point A and choose the menu item *Transform/Mark Center*. Select only segment m and point B and choose *Transform/Rotate by* 90 degrees. While the segment and point is still selected, choose *Transform/Dilate* by a fixed ratio of 2 to 1, placing 2 in the numerator and 1 in the denominator. (You should recognize that you just rotated segment m and then doubled its length.) Now select only point B and *Transform/Mark Center*. Select only segment m and point A and *Transform/Rotate by* -90 degrees. While the segment and point is still selected, choose *Transform/Dilate* by a fixed ratio of 2 to 1. Your figure should look similar to

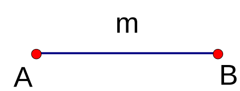


(but without labels). Construct a line segment from C to D. Hide points E and F. Select, in order, points A, B, C, D and segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{AD}$. With these objects highlighted, click and hold on the Custom Tool (the button beneath the Text tool), and choose 'Create New Tool'. (Throughout this lab, you will often need to click and hold down the mouse button to see the Custom Tool menu options; just clicking the tool will not always display the list of options.) In the dialog box that appears, type *Make Rectangle* for the tool name, then click OK. Click on the Custom Tool button and select your homemade rectangle-making button. Your arrow cursor should have a point stuck to it. Click, drag and let go! You should have just made a rectangle, in a similar way to how you constructed circles with the circle tool! To practice using this tool, make a lot of rectangles on your sketch – having different sizes and aiming in different directions. (Note that these rectangles all share the characteristic that the longer side is twice as long as the shorter side, due to the dilation ratio we used. This will suffice for now.) Select all objects on your sketch and cut them – a fast way to do this is *Edit/Select All*, then *Edit/Cut*. (Basically, you want a clean sketch but you don't want to lose the tool you just created, so don't choose *File/New Sketch*.)

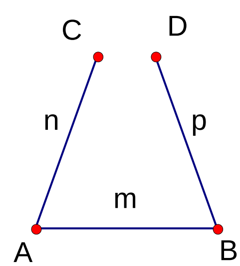
Note: If you want to delete or rename a custom tool, click and hold on the Custom Tool button and choose Tool Options.

Creating a Tool to Make Isosceles Triangles (that are not equilateral)

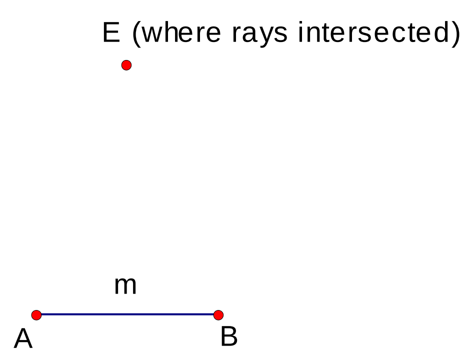
Create a horizontal line segment about one inch long. As mentioned above, you do not need to label the segment or points. You have created a segment such as



Select only point A and *Transform/Mark Center*. Select only segment m and point B and *Transform/Rotate by* 70 degrees. Select only point B and *Transform/Mark Center*. Select only segment m and point A and *Transform/Rotate by* -70 degrees. Your figure should look similar to



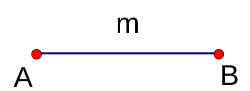
(but without labels). Select only point A and point C and *Construct/Ray*. Select only point B and point D and *Construct/Ray*. Select both rays (click high enough up on the ray to avoid clicking only the little segments n or p) and *Construct/Intersection*. Hide the following: the two rays, points C and D, segments n and p. Your figure should look similar to



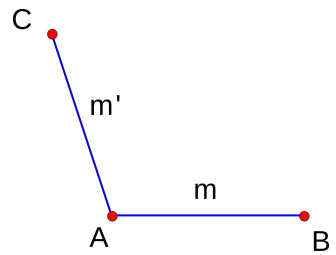
Connect vertices A and E with a line segment and do the same for B and E. Select, in order, vertices A, B, E and segment m and the other two segments. With these objects highlighted, click on the Custom Tool, and choose 'Create New Tool'. Name this tool *Make Isosceles Triangle*. Click on the Custom Tool button and select your homemade triangle-making button. Your arrow cursor should have a point stuck to it. Click, drag and let go! You should have just made an isosceles triangle! Practice this tool by making a lot of triangles on your sketch – having different sizes and pointing in different directions. (Note that these triangles all share the characteristic of having two 70 degree angles, due to our 70 degree angle constructions. This will suffice for now.) Make a clean sketch again by cutting all objects. You will not lose the custom tools you created.

Creating a Tool to Make Regular n-gons

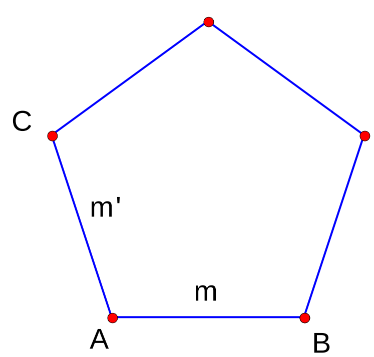
Create a horizontal line segment about one inch long.



Select only point A and *Transform/Mark Center*. Select only segment m and point B and *Transform/Rotate by* ____ degrees. How many degrees should the segment be rotated? It depends on how many sides the regular polygon has. Recall that in a regular n-gon, $m(\text{vertex} <) = \frac{(n-2)\times 180^\circ}{n}$. The following instructions will help you create a custom tool for making regular pentagons. Afterward, you will need to replicate these steps to make four other regular n-gons on your own. You know that the measure of a vertex angle in a regular pentagon is $\frac{(5-2)\times 180^\circ}{5} = 108^\circ$. Therefore, when you selected segment m and point B, you should have rotated it by 108° to achieve:



Mark point C as center of rotation and rotate segment m' and point A by 108°. Continue this process to achieve:



Select the vertices in order and the segments in order, click on the Custom Tool, and choose 'Create New Tool'. Name this tool *Make Regular Pentagon*. Test your new tool and then clear the page again.

Follow this similar procedure again to create custom tools titled *Make Equilateral Triangle*, *Make Square*, *Make Regular Hexagon*, and *Make Regular Octagon*. You have now created a total of seven new custom tools to use for your net designs.

Making Nets

As you discovered in class with the plastic *Polydron* manipulative for creating nets of polyhedra, three-dimensional shapes can have more than one two-dimensional net. Therefore, in *Sketchpad*, use your newly created custom tools to make the listed amount of distinct nets for each of the solids below:

Right hexagonal pyramid	- **three distinct nets**
Right octagonal prism	- **three distinct nets**
Tetrahedron	- **two distinct nets**
Cube (hexahedron)	- **three distinct nets**
Octahedron	- **three distinct nets**
Dodecahedron	- **one net**
Icosahedron	- **one net**

Using *Sketchpad's* text-writing tool, label each net with the name of its corresponding three-dimensional shape. You should have created a total of 16 labeled nets.

Recall from classroom discussion that the latter five shapes in the list above are the **Platonic solids**, the only five regular convex polyhedra.

Based on the three-dimensional shapes that your nets create, complete the attached table related to Euler's formula.

Hand in:

1) a copy of your *Sketchpad* file so your tools can be tested,
2) printouts of the 16 labeled nets, and
3) the completed worksheet related to Euler's formula.

*** <u>Also</u>, print one net of each shape for yourself. Cut out the net, fold appropriately and tape or glue it into its respective three-dimensional shape. (Note that when you cut out a net, it is useful to leave some excess "edge" to be used for tabs to aid in the taping/gluing process). Each of the seven solids should be labeled with both your name and the shape's name, and handed in along with the other materials listed in 1) – 3) above.

Name ____

Exploring Euler's Formula for Polyhedra

1. Using your three-dimensional shapes created from your two-dimensional nets, complete the table below.

Name of Polyhedron	# of Faces, F	# of Vertices, V	# of Edges, E	F + V =	E =
Right hexagonal pyramid					
Right octagonal prism					
Tetrahedron					
Cube (hexahedron)					
Octahedron					
Dodecahedron					
Icosahedron					

2. Based on your results in the last two columns of the above table, make a conjecture for the equation known as "Euler's formula", or "Euler's equation", for polyhedra.