Mathematical Modeling: Some Ideas and Suggestions for Pre-service Teacher Preparation

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Abstract

The challenges of preparing secondary level math teachers for a new century of instruction have taken us well beyond the "standards" efforts of the past two decades. Reasoning, proof, and the development of new ways to represent mathematical ideas have begun to emerge as new basics. Mathematical modeling activities allow students to take information they have already learned and apply it in process of scientific inquiry thus allowing them to touch on the basic concepts of proof in a controlled environment. A sample modeling activity with a possible solution for mathematically telling time has been included.

Introduction

An important aspect of the continually changing reform movement in secondary level mathematics education is that the next generation of teachers be able to shape the mathematics curriculum according to the future needs of their students. Of course, this is a difficult task given that we currently don't know what those needs are. It is therefore most important that these teachers of mathematics continue to grow with respect to the pedagogical techniques that have the greatest classroom potential. Researching these techniques, and making them usable for the classroom requires a great deal of effort, but good teachers would certainly agree that the resources they bring to bear on behalf of their students set a foundation of success or failure for those students.

The reform efforts of the past decade have resulted in a mass of professional documents, curriculum standards, and reports, all of which are intended to strengthen a new teacher's repertoire of techniques as they enter their first classroom. Yet with all of the various forms of assistance, the mathematics classrooms of the 21st century will probably look very similar to those that have been so common for the past 50 years. The fact that we know so much more now than we did 50 years ago, at least from a scientific standpoint, has appeared to have very little impact on what or how things are taught in the secondary math classroom. Agreed, infusing technology has brought flavor to the mathematics classroom, but the textbooks along with their very familiar format still seem to be the preferred method of instruction.

So if textbook based instruction is not doing an adequate job of preparing a mathematically literate society, what will? One possibility may be Instructional activities using a mathematical modeling approach. This type of instruction has proven to be both effective and engaging for students at the Jr. high and high school levels. This philosophy of instruction also provides an opportunity for some of the most valuable curriculum topics in today's mathematics classrooms to be revisited in the context of an interactive practical applications format that preempts the "what do we need this for" question. The mathematical modeling approach to instruction is indeed "front heavy" for new teachers, but allows for the kind of valuable exploration in mathematics that has been largely absent to date.

What is Mathematical Modeling

Mathematical modeling can perhaps be best defined as "the process of scientific inquiry" for mathematics. The scientific method is obviously a comfortable mode for teachers of science, but is rarely seen in mathematics classrooms. Students engaged in mathematical modeling activities spend the majority of time experimenting in applied physical situations in an attempt to find patterns and consistencies in sets of data. These data sets could come from a number of different sources including those already existing in different forms on the Internet, those in books and newspapers, or they could even be collected as part of a classroom activity. Part of the impetus for mathematical modeling activities in the classroom is to help students understand that mathematics is not a discipline where complex solutions to problems are innately obvious or solvable in a matter of just a few minutes. In fact, any good mathematical modeling activity should be appropriately vague so students don't get the impression that the activity is just another textbook assignment. The problems are designed to be difficult so that they better represent the scope of mathematics used for solving real life problems. However, with these more difficult problems, correspondingly greater amounts of time must be allotted for students to investigate the topics and develop solutions. This is an area where undergraduate teacher preparation programs can help pre-service teachers of mathematics refocus on instructional techniques that embrace the less traditional classroom methods of problem solving and modeling.

Why Mathematical Modeling…. Blooms Taxonomy: Old Topic, New Idea

Mathematical modeling, as a formal part of the mathematics curriculum, can be difficult to define and conceive; however, a popular topic from pre-service teacher education training may be helpful. One of the primary guiding principles related to lesson planning and questioning in undergraduate teacher preparation is Bloom's Taxonomy. The six levels of Bloom's Taxonomy (knowledge, comprehension, application, analysis, synthesis, and evaluation) are typically used to assist teachers in defining levels of cognitive processing for their lessons. Although the topic of Bloom's Taxonomy is not a new one, a re-conceptualization of these six levels may help to put mathematics instruction in perspective as per the NCTM Standards (1989).

It is not unusual to see classroom teachers justifying lesson objectives by suggesting that they be designed to have students think at a given level. For instance, adding two fractions taken from student made measurements might mean they are operating at the "application" level. Yet, there still appears to be confusion about what each level actually is. The lists of synonyms traditionally used to define the six levels don't always provide a clear understanding of how to write objectives that focus on different levels for a single mathematical topic. In fact, very few mathematics textbook problems go past the application level, which is why many teachers of mathematics find it difficult to stretch their objectives and corresponding lessons to fit higher levels of cognition. Even the textbook “problem solving” applications do not address the higher levels of Bloom’s Taxonomy in the way the NCTM originally intended because the solutions to those problems typically follow a set of closely related practice exercises. In revisiting the three higher levels of Bloom’s Taxonomy (analysis, synthesis, and evaluation), we can provide for a way to rethink the NCTM Standards related to Problem Solving, Communication, Reasoning, and Connections, and by extension, mathematical modeling. In rethinking the Analysis and Synthesis levels of the taxonomy, one might find that they are appropriate levels of cognition for true problem solving activities in the way the NCTM originally intended.

Problem Solving in its true nature assumes that the solution is not innately obvious, and that problem solvers need to follow a more general heuristic in order to come to a conclusion. There may also be numerous solutions to one problem, which again indicates that any specific algorithmic application is at a lower level of cognitive processing. The highest level of Bloom’s Taxonomy (evaluation) is perhaps the most difficult to define. Evaluation based objectives have typically done little more than superficially ask the question “Is this any good?” This is where mathematical modeling comes to the proverbial rescue. Math modeling is a multi-faceted process of *problem solving* (NCTM Standard #1) combined with stages of continual refinement. Evaluation as it pertains to mathematical modeling presumes a cyclical process whereby the student solves a problem to a reasonable extent using topical information from the classroom instruction, and then reexamines data to search for a way to make the solution better, faster, or more efficient. In essence, it is a continued *reasoning* (NCTM Standard#3) process that takes place even after an initial solution is reached. Students participating in mathematical modeling activities will be exposed to many different levels of “thinking mathematically.” The application of specific mathematical processes is important but is secondary to the synthesis process, where they bring several algorithmic applications together to propose a solution to a broader problem. The model can then be modified and refined as the students discover, or are instructed in, better ways to approach the problem.

The teacher designing a modeling activity has the difficult task of articulating the problem in such a way as to provide clarity without being too prescriptive. This is done to emphasize that mathematical modeling is a process of continual refinement and modification. Instructional considerations that may be helpful in using mathematical modeling activities in the classroom are as follows:

1. Students have some control over how they approach a problem. This is not typically the case with most textbook problems.
2. Good modeling activities are easily scalable to different grade levels.
3. Problem solving and mathematical modeling are different processes.

Problem solving typically acts as a process oriented approach whereby students find a specific solution to a specific problem. Mathematical Modeling is an experimental approach where a problem is solved and continually refined over time in order to be more efficient, faster, or more accurate. Problem solving in many cases has a solution that is either correct or incorrect. Mathematical modeling is a process where few answers are incorrect, they just require continual revision.

4. Mathematical modeling focuses primarily on the “general case.” Students must at least generally understand the concept of a variable, which is why modeling activities below the Jr. High school level are difficult to construct.
5. Mathematical modeling activities are difficult to assess. An elegant solution may be an approach that works coincidentally but a student can justify why. Another solution to the same problem may utilize some specific procedure from the textbook, yet the student has no understanding of why they chose that method nor why it works.

The premise of the mathematical modeling concept is not that the traditional courses in the curriculum need to be replaced, but rather accented in the appropriate spots to better emphasize the practical use of the concepts we do teach. Because mathematical models can take on many forms, the processes by which problems are approached are numerous and varied. Some of the more basic modeling structures lend themselves very well to established secondary level curriculum (i.e. numerical tables and patterns, graphs, systems of equations, etc.). Others may be more algorithm-based problems that require a computer or graphing calculator as an extension. Although no one set of rules is inherent to all mathematical modeling activities, the following set of steps can act as general guidelines for students engaged in mathematical modeling activities:

1. Define the problem.
2. Establish the factors that affect the outcome.
3. Define which of these factors are outcome variables.
4. Establish a relationship between the defined variables to derive a formula or alternately defined model or algorithm.
5. Test the model with known values from previously collected data.
6. Refine the model for accuracy and efficiency.

At this point, a sample modeling activity has been constructed with a possible solution. The activity is designed for use in a university level methods class for prospective secondary mathematics teachers. This model has been included to help demonstrate the dynamics of a secondary level classroom modeling lesson.

Example Modeling Activity: Mathematically Telling Time

With all of the year 2000 uncertainties now past, it may seem a bit anticlimactic to talk about the impact that time has on our day to day lives, but telling time has been essential to human kind ever since we have had the desire to maintain a schedule. There are probably more catch phrases relating to time than there are phrases pertaining to any other topic. Time is of the essence, time is up, there is no time like the present, timing is everything, just in time, day time, night time, dinner time, party time... you get the idea. It is even fair to say that this preoccupation with time led early watch makers, and now scientists to spend a great deal of *time* and energy building clocks that boast tremendous accuracy. In fact atomic clocks (using the mathematically predictable resonance frequencies of elements like Cesium) have been said to be accurate to within trillionths of a second. Such precision in the measurement of time has paved the way for other scientific discoveries as well.

However, history has shown that utility has not been society's only interest with time pieces. We are typically just as impressed with their beauty, elegance, and refinement as we are with their accuracy. Yet with all gilded and ornate, not to mention technologically sophisticated clocks in existence, my favorite is still the sundial. There are many different kinds of sundials, most of which have no moving parts, so they can literally work forever. For many, a sundial's beauty and only true source of sophistication resides in pure mathematics. Deriving the appropriate formulas and relationships allow us to predict how shadows fall on an object relative to the sun's apparent movement over the surface of the earth, and these shadows are the only "moving" parts on a sundial. Also, because the mechanical interplay between the parts of a sundial are minimal to nonexistent, building one is relatively easy once we have produced the mathematical model needed for our particular sundial design. Our task then is to find a mathematical way to use shadows to tell time

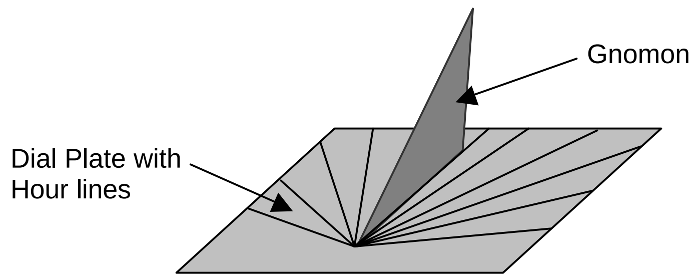
The Mathematical Model: One Possible Solution

Of course, the mathematical model is typically the most difficult part of a complex problem like making a sundial work. There are many different kinds of sundials, so one would assume that there are also many different mathematical approaches to building one. Although mathematical modeling has not traditionally been a popular approach to secondary mathematics instruction, it does allow for a more realistic view of how mathematicians and scientists approach real problems. It was this realization that inspired the sundial problem as a classroom activity.

Building a Sundial - Some Background:

One of the more common types of sundial is the *horizontal sundial,* which is most appropriate for this activity. The horizontal sundial consists of a flat surface, known as the dial plate, which rests parallel to the ground. The next piece is called the gnomon (the word gnomon is derived from a Greek word meaning "one who knows"), which is a roughly triangular plate that sticks up out of the dial plate like a shark fin. The final component is the upper slanting line of the gnomon and is called the style. A horizontal sundial is illustrated in figure 1.

Figure 1: Horizontal Sundial

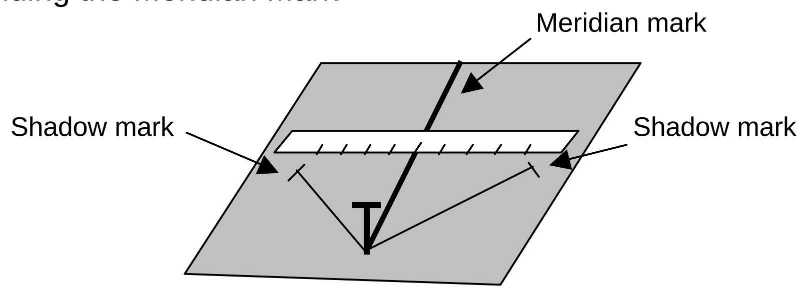


This particular sundial design has three distinct phases of construction: finding the meridian mark (which represents local apparent noon), angling the gnomon, and marking the dial plate. Each of the phases requires an increasingly more sophisticated understanding of geometry and trigonometry, so it really creates three unique classroom problems, or more appropriately, three modeling activities. The key to constructing a horizontal sundial is to understand that by using the patterned apparent movement of the sun over the surface of the earth, we can follow the path of a shadow created by a vertical object. Because of the east to west movement of the sun, shadows are long and pointing westerly in the morning when the sun rises, short as noontime approaches, and long and pointing easterly as the sun sets. Classroom discussions related to this phenomenon help emphasize the need for a mathematical approach to this problem. It is very important to remember that the following line of thinking and the resulting mathematical model represents one of many possible solutions.

Step 1: Finding the meridian mark

The meridian lines are imaginary lines running along the earth's surface from the north to the south poles. Our own meridian line is then nothing more than the line passing through the spot where we happen to be. To find this line without special instrumentation, we need the assistance of a shadow. Because a vertical object is needed to create a shadow, and we want a shadow length that is easy to measure, a nail pounded vertically into a rectangular board can be used as a center mark. Local apparent noon is the point at which the sun is directly overhead to the observer. Therefore, if we follow the path of the shadow created by the nail, we should be able to find local apparent noon or the meridian mark. Concentric circles drawn around the nail can be used to measure the shadow in both the morning and the afternoon. If the tip of the shadow is measured in the morning at a given length, and then again in the afternoon when it is the same length, an angle is created between the morning shadow mark, the base of the nail, and the afternoon shadow mark. Using the base of the nail as the vertex of the angle and then bisecting the angle, we can create a line that represents the noon mark. This is illustrated in Figure 2.

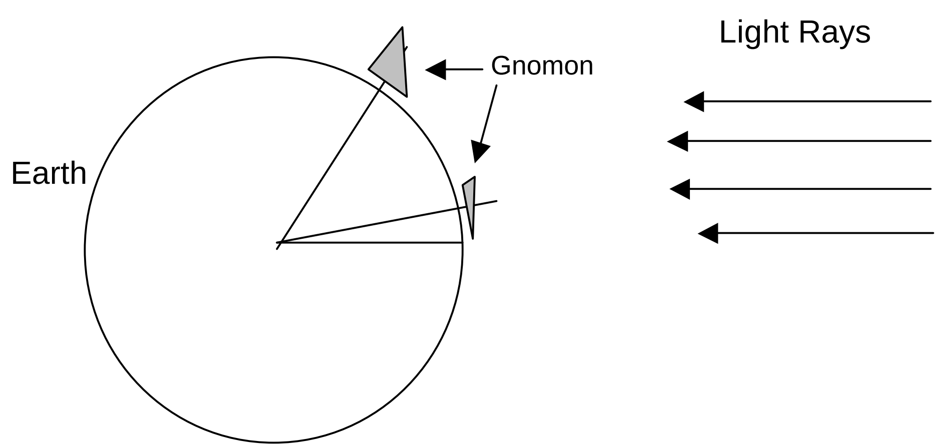
Figure 2: Finding the Meridian Mark



Step 2: Angling the gnomon

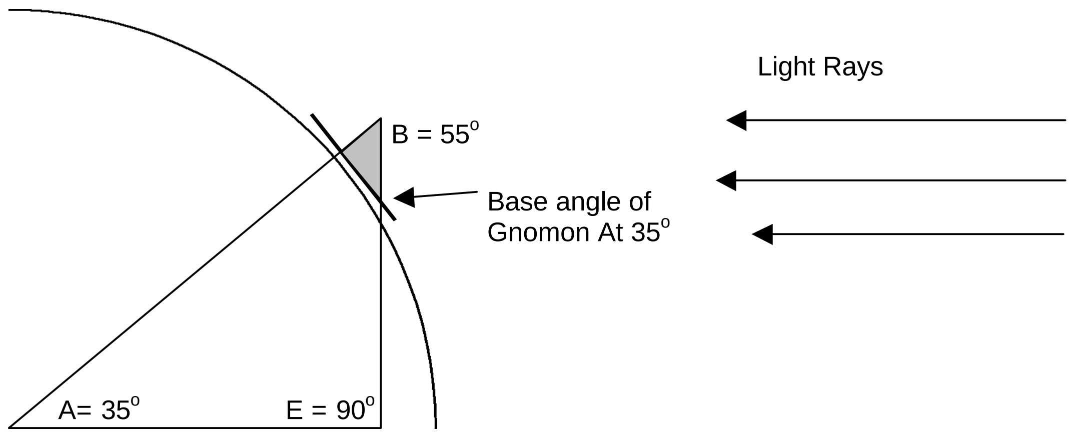
Assuming that the meridian mark has been established using an effective model, the next step is to angle the gnomon. Students must be sure that the dial plate is sitting flat on the ground, because we need to angle the gnomon such that the sun's rays hit it at a right angle. This means that the gnomon must be angled differently at different latitudes. Through classroom discussion, research, and some experimentation students slowly realize this fact. Interestingly enough, students also discover that most commercially produced sundials will not work everywhere because the gnomon is angled incorrectly for most latitudes. For instance, a sundial used in North Dakota would have a gnomon with a much greater angle than one used in southern Texas. So, once again, the need for experimental mathematics becomes more clear. Figure 3 illustrates how the gnomon must be angled at different latitudes.

Figure 3: Sun's Rays Striking the Gnomon



Using the fundamentals of similar triangles and some simple angle computations, students can prove that the gnomon must have a base angle that is equivalent to the latitude where the sundial rests. For instance, if the sundial sits at 35^o north latitude, then the base angle of the gnomon must also be 35^o. Figure 4 illustrates how the gnomon angle is determined by the latitude.

Figure 4: Calculating the Gnomon Angle

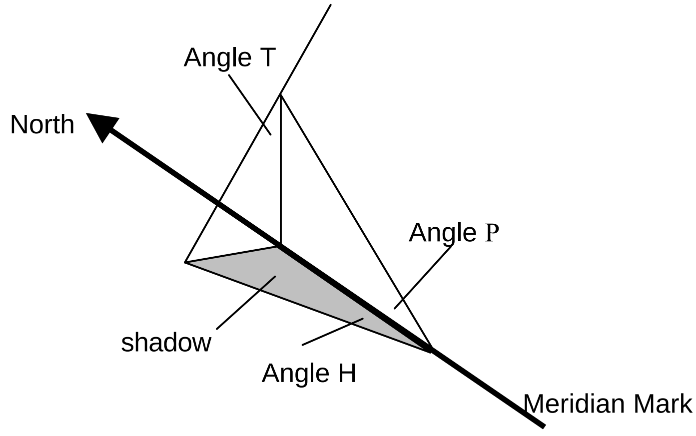


If a latitude of 35^o is used, the radius extending from the center of the earth (point A) intersects the dial plate at a right angle. This means that the gnomon itself could easily be a right triangle, as shown above. Extending the style line from point B to point E creates a right triangle similar to that of the gnomon. Therefore, the base angle of the gnomon must be congruent to the angle representing latitude.

Step 3: Marking the dial plate

This is mathematically the most difficult model needed for building the sundial. Marking the dial plate essentially means putting on the hour marks. Because the earth rotates through a central angle of 360^o in 24 hours, we can assume that each hour is defined by a 15^o arc of the Sun's apparent movement over the surface of the earth. This is true at any latitude. Figure 5 illustrates the relationship between the gnomon, the dial plate, and the shadow. Once again, the following model represents one of many possible models that students could derive.

Figure 5: Marking the Dial Plate



We will call angle T the angle determined by the top of the gnomon and the sun's rays. We have already determined that an angle of 15^o represents one hour away from the noon mark. So at 11:00, angle T is 15^o, at 10:00, angle T is 30^o, and so on. If the length of the style is known, we can calculate the length of the vertical side of the gnomon by the following formula: Height = sin P(S), where S is the length of the style. If we now want to mark the 11:00 hour on the dial plate, angle H, which is the base angle of the triangle created by the shadow, can be calculated using the length of the shadow opposite angle H: angle H = (tan T)(sin P)S. Given some time, students ultimately derive the following formula: tan H = (tan T)(sin P). Knowing this relationship now allows them to put the hour marks on the dial plate.

For the sake of easy test calculations, suppose that we are at a latitude of 30^o and we have constructed our style to be 1 foot long. Note however that the length of the style will have no effect on the outcome of angle H. The following substitutions can be made:

tan H = (tan T)(sin P)
tan H = (tan 15)(sin 30)
tan H = (.268)(.500)
tan H = (.134)
Therefore angle H = 7.63^o

So the 11:00 hour line would lie at an angle of 7.63^o to the left of the meridian mark. Also, because the hour marks are symmetrical with respect to the noon mark, 1:00 would be 7.63^o to the right of the noon mark. Continuing in this fashion would allow us to accurately mark all of the daylight hours on the dial plate. Students would also be able to mark the dial plate without using a formula by using a regular clock and marking the point where the shadow was at each hour. This would allow them to check the accuracy of the model, and make any modifications if necessary.

Summary

Although this particular mathematical model doesn't account for some things like daylight savings time or the change in relative tilt of the earth by seasons, it certainly illustrates the value of mathematically modeling patterns for the purpose of solving problems. This model would also be revisited and perhaps modified to be more accurate at some point in the future. The sundial is of course a limited time piece since it can only be used when there is daylight, but it still remains a symbol of humankind's ingenuity. Perspective teachers still need to see plenty of examples until they get to the point where they can generate modeling activities on their own, but by participating in these types of lessons, they see first hand how valuable they can be.

High school students will probably never see this kind of question on the SAT, but this general type of problem appears to be a very rich area of investigation into patterned mathematics for the secondary classroom. Students allowed to build mathematical models and test them on real life data tend to have a much more intimate knowledge of the processes and procedures that we so desperately try to get them to understand by reading the textbook. However, we still spend inordinate amounts of time in new teacher preparation programs on more “old fashion” instruction involving textbook procedures.

Regularly demonstrating concepts through classroom activities, similar to this one, also helps new teachers of mathematics learn to approach the instruction of mathematics from a more scientific standpoint, where discovery is the bottom line rather than what appears on the test answer key. Finally, the age old question, “what are we ever going to use this for?” can be answered quite simply, “Ancient civilizations used these concepts to help them design and build a clock called a sundial, and this is how they did it ..... And just like these people, how you are going to use it is limited only by your curiosity and imagination.”